Fractional Order State Space Canonical Model Identification Using Fractional Order Information Filter

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Abstract—In the present paper the identification and estimation problem of a fractional order state space system will be addressed. This paper presents a fractional order information filter and also a hierarchical identification algorithm to identify and estimate parameters and states of a fractional order system. Then, merging this algorithm with fractional order information filter, a novel identification method based on hierarchical identification theory is introduced to reduce the computational complexity. Finally, the applicability and performance of this platform on an exemplary system is examined.

Index Terms—Fractional Order Systems, Fractional Order information filter, Recursive Identification, Hierarchical Identification Principle

I. INTRODUCTION

In 1695, the concept of fractional calculus was expressed for the first time by Leibniz and L'Hospital. In the late nineteenth century, Riemann and Liouville give the first definition of fractional derivatives. In recent years, because of their many applications, fractional-order systems (systems that contain fractional derivatives and fractional integrals) have received the attention of many researchers. Fractional order calculus is able to define many complex systems that are not easily represented applying conventional integer order derivative or integration [1]. Application-wise, fractional order calculus is used in various fields such as adaptive systems design [2, 3], electro-magnetics[4], signal processing[5, 6], robust controller design for wind turbine generators[7], etc. More applications for this theory is now available in literature such as mass and heat transformation, electro-chemical processes, dielectric polarization, electric power distribution lines, visco-elastic materials such as polymers and plastics, flexible structures, traffic in data networks, and biological systems[8-12]. Fractional-order system identification methods can be broadly classified as techniques in the frequency domain and in the time domain. Due to their long-term memory behaviour, the identification of fractional-order models are more difficult than for integer-order models; therefore, different algorithms have been proposed in the frequency domain to solve this problem. In recent years many investigations are performed in this field [13-20], but there are no reports on recursive identification of fractional order state space system parameters. The hierarchical identification approach proposed in this paper has less computational overhead compared to the previous methods. The basic idea is to use the hierarchical identification principle to decompose the system model into several sub-models with smaller dimensions and fewer variables, and then to identify the parameter vector of each sub-model. The identification error converges to zero as shown in [21]. The proposed hierarchical identification method has many applications in various fields such as system modeling, signal filtering and adaptive control. In addition, another application is in state feedback in which the states are used for stabilization of unstable dynamical systems. Thus, considering the importance of state estimation, in cases where the parameters of a fractional-order state-space system are unknown as well as the state variables, this identification method is very useful. The hierarchical identification method provided in this paper, reduces the computational burden and the parameter estimation error given by the proposed algorithm converges to zero. In this paper, a canonical state space model has been used for identification of fractional order systems for the first time. This property can be exploited in system identification because the canonical models have a special form that makes them identifiable. Furthermore, the recursive least square approach that is proposed in this paper for fractional order system identification is a novel approach that has not been proposed previously for fractional order canonical state space model. In addition, although several papers can be found in the literature relating to the identification of fractional order systems, but the new hierarchical identification method that is proposed in this paper reduces the computational burden and is easy to implement on a computer, so that it will significantly reduce the execution time (approximately half). The advantage of using the state space model is that it can be easily expanded to multi-input multi-output (MIMO) systems, while the identification methods that are based on a transfer function cannot be easily expanded to MIMO systems. The standard Kalman filter equations require the inversion of a \( r \times r \) matrix, (where \( r \) is the number of measurements). The information filter equations require at least a couple of \( n \times n \) matrix inversions (where \( n \) is the number of states). So if the number of measurements is much greater than the number of states \((r >> n)\) it may be computationally more efficient to use the information filter[22]. Furthermore, if \( R_i = R \) has a fixed value, it doesn’t require the inversion of a \( r \times r \) matrix.
at each stage, so using the information filter reduces the computational burden.

This paper is organized as follows; in section II the issue that is going to be solved is more briefed. In section III the identification problem and in section IV the state estimation problem are addressed. Then, the concurrent identification and estimation is introduced in section V. In section VI simulation results on an exemplary system are presented which are followed by conclusions and remarks in section VII.

II. PROBLEM FORMULATION

Suppose a fractional order discrete time linear stochastic state space system as follows [23]:
\[
\Delta^\gamma X(k+1) = AX(k) + Bu(k) + w(k)
\]
(1)
\[
X(k+1) = \Delta^\gamma X(k+1) - \sum_{j=0}^{k} \Delta^{\gamma}Y_j X(k+1 - j)
\]
(2)
\[
Y(k) = HX(k) + v(k)
\]
(3)
where \( Y \) is the order of the fractional difference \( Y \in \mathbb{R}^\gamma \) and \( X(k) \) is the state vector \( X(k) \in \mathbb{R}^n \). \( w(k) \) and \( v(k) \) are the process and measurement noises, respectively. Furthermore, \( \gamma \) is defined as:
\[
\gamma = \Gamma(Y + 1) / \Gamma(Y - j + 1)
\]
(4)
Assuming that \( \gamma > 0 \), Euler’s function \( \Gamma \) is defined as:
\[
\Gamma(x) = \int_0^\infty e^{-v}v^{x-1}dv
\]
(5)
\[
\Delta^\gamma X(k+1) = \begin{bmatrix}
\Delta^{\gamma}X_1(k+1) \\
\vdots \\
\Delta^{\gamma}X_n(k+1)
\end{bmatrix}
\]
(6)
where \( \gamma_1, ..., \gamma_n \) are the order of the system equation and \( n \) is the number of system equations.

**Assumption 1**: \( v(k) \) and \( w(k) \) are two independent white noises with zero mean and covariance matrices \( R(k) \) and \( Q(k) \), respectively. In other words, we have:
\[
E[w(k)] = 0, E[v(k)] = 0
\]
\[
E[w(k)w^T(j)] = Q(k)\delta(k - j)
\]
\[
E[v(k)v^T(j)] = R(k)\delta(k - j)
\]
\[
E[w(k)v^T(j)] = 0,\forall k, j
\]
\[
E[X(0)] = \bar{X}(0)
\]
\[
E[(X(0) - \bar{X}(0))(X(0) - \bar{X}(0))^T] = P_0
\]
And also, \( X(0) \) is uncorrelated with \( v(k) \) and \( w(k) \). Also
\[
Y(k) = \begin{bmatrix}
y_1(k) \\
y_2(k) \\
\vdots \\
y_n(k)
\end{bmatrix}
\]

Merging equations Eq.1 and Eq.2 will result into the following equations:
\[
X(k+1) = AX(k) + Bu(k) + w(k) + \sum_{j=1}^{k+1} C_j X(k+1 - j)
\]
(7)
\[
C_j = (-1)^{j+1} \text{diag} \left[ \begin{array}{c} 
y_1 \\
y_2 \\
\vdots \\
y_n
\end{array} \right]
\]
(8)
Where the values and orders of \( A, B, \) and the state vector \( X(k) \) are defined as follows:
\[
A = \begin{bmatrix}
b_1 \\
b_2 \\
\vdots \\
b_n
\end{bmatrix}, B = \begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_n
\end{bmatrix}, X = \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix}
\]
\[
C_j = \begin{bmatrix}
c_{11,j} & 0 & \cdots & 0 \\
c_{22,j} & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots \\
0 & \cdots & \ddots & \ddots
\end{bmatrix}
\]
(9)
\[
H = \begin{bmatrix}
\alpha_{11} & \cdots & \cdots & \cdots \\
0 & \alpha_{22} & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \ddots & \ddots
\end{bmatrix}
\]
(10)
In addition, we also assume that:
\[
u(k) = 0, k \leq 0
\]
\[
Y(k) = 0, k \leq 0
\]
\[
X(k) = 0, k \leq 0
\]

III. THE MODEL USED FOR IDENTIFICATION

The goal in this part is to propose a recursive algorithm for parameter identification in fractional order state space system model. To propose such algorithm, a linear regression equation is necessary to be formed. Since the state space model of the system is concerned rather than its transfer function, forming such equation is more difficult. Therefore, a hierarchical method is applied to have a proper form of the regression equation in recursive equation.

Expanding Eq.7 gives:
\[
X(k+1) = AX(k) + Bu(k) + C_1 X(k) + C_2 X(k-1) + \cdots + C_{n-1} X(k-n-1) + C_n X(0)
\]
(11)

In addition, by opening Eq. 9, we obtain:
\[
x_1(k+1) = x_1(k) + b_1u(k) + c_{11}x_1(k) + \cdots + c_{11,k+1}x_1(k) + \cdots + c_{11,k+1}x_1(k)
\]
\[
x_2(k+1) = x_2(k) + b_2u(k) + c_{22}x_2(k) + \cdots + c_{22,k+1}x_2(k) + \cdots + c_{22,k+1}x_2(k)
\]
\[
x_{n-1}(k+1) = x_{n-1}(k) + b_{n-1}u(k) + c_{n-1}(k) + \cdots + c_{n-1,k+1}x_{n-1}(k) + \cdots + c_{n-1,k+1}x_{n-1}(k)
\]
\[
x_n(k+1) = a_1x_1(k) + a_2x_2(k) + \cdots + a_nx_n(k) + b_nu(k) + c_{m,2}x_2(k) + \cdots + c_{m,2}x_2(k) + \cdots + c_{m,2}x_2(k)
\]
(12)
To form the above stated equations into a suitable form for regression equation, we define the shift operator as $z^{-1} X(k) = X(k-1)$. Shifting the state space equations introduced in 10 as follows we get:

$$(z^{-1}) x_i(k+1) = (z^{-1}) x_i(k) + b_i u(k) + c_{i1} x_1(k-1) + \ldots + c_{i,k} x_k(0)$$

$$x_1(k+1) = x_1(k) + b_1 u(k) + c_{11} x_1(k-1) + \ldots + c_{1,k} x_k(0)$$

$$x_2(k+1) = x_2(k) + b_2 u(k) + c_{21} x_1(k-1) + \ldots + c_{2,k} x_k(0)$$

$$\vdots$$

$$x_{n-1}(k+1) = x_{n-1}(k) + b_{n-1} u(k) + c_{n-1,1} x_1(k-1) + \ldots + c_{n-1,k} x_k(0)$$

$$x_{n}(k+1) = x_{n}(k) + b_{n} u(k) + c_{n,1} x_1(k-1) + \ldots + c_{n,k} x_k(0)$$

Or

$$x_i(k) = x_i(k-1) + b_i u(k-1) + c_{i1} x_1(k-1) + \ldots + c_{i,k} x_k(k-1)$$

As a result, it can be concluded that combining the equations in 11 a regression equation will be resulted:

$$y_i(k) = x_i(k) + v(k) = \phi^T(k) \Theta + N(k) + v_i(k)$$

where

$$\Theta = [a_1 \ a_2 \ \ldots \ a_n \ b_1 \ b_2 \ \ldots \ b_l]^T$$

$$\phi(k) = [x_1(k-n) \ x_2(k-n) \ \ldots \ x_k(k-n)]^T$$

$$N(k) = c_{11} x_1(k-1) + \ldots + c_{1,k} x_k(0) + c_{21} x_1(k-2) + \ldots + c_{2,k} x_k(0)$$

$$+ c_{n-1,1} x_1(k-n-1) + \ldots + c_{n-1,k} x_k(k-1) + c_{n,1} x_1(k-n) + \ldots + c_{n,k} x_k(k-n)$$

In $N(k) \ , \phi(k)$ the coefficients $c_{11,1}, \ldots, c_{n,k,1}$ are known and the states are estimated using the fractional-order Kalman filter. For this reason, $N(k)$ can be considered as an additive term in the regression equation.

Based on the available regression equation in Eq.12, a recursive algorithm to identify the parameters of fractional order state space system seems necessary. Based on what discussed in [24], to get the optimal estimation the following objective function have to minimized.

$$\hat{\Theta}_{k+1} = \arg \min_\Theta \frac{1}{\delta} \sum_{i=1}^{k} (y_i(i) - \hat{y}_i(i, \Theta))^2$$

Moreover, to ensure that the system has the identifiability property, the $[\sum_{i=1}^{k} \phi(i-1) \phi^T(i-1)]^{-1}$ matrix must exist.

$$\hat{\theta}_i = \sum_{i=1}^{k} \phi(i-1) \phi^T(i-1) \sum_{i=1}^{k} \phi(i-1)y_i(i)$$

The recursive least squares identification algorithm is as follows [25]:

$$\hat{\Theta}_{k+1} = \hat{\Theta}_k + F_k \Phi(k) \varepsilon(k+1)$$

$$F_{k+1} = F_k - F_k \Phi(k) \Phi^T(k) F_k$$

$$\varepsilon(k+1) = y_i(k+1) - \hat{\Theta}_k \Phi(k) - N(k)$$

Here, $F_k$ is the adaptation gain matrix, and its initial value is equal to:

$$F_0 = \frac{1}{\delta} I, \ 0 < \delta < 1$$

IV. THE INFORMATION FILTER ALGORITHM

The regression equation stated in 12 consists of $\phi(k), \Theta$ and $N(k)$ vectors and it also consists of different states with distinct time and coefficients. In contrast to ordinary regression equations where $\Theta$ includes vector of unknowns and $\phi(k)$ includes the given data, here both vectors consist of a set of unknowns. As $\phi(k)$ and $N(k)$ consists of states of the system, state estimation seems to be necessary to resolve the above issue. Here, a novel fractional information filter will be proposed for state estimation in this system.

As expressed in the Kalman filter, $(P)$ represents the uncertainty in the state estimate. Therefore; in the limit, as $P \rightarrow 0$ we have perfect knowledge of $X \ , \$ and as $P \rightarrow \infty$ we have zero knowledge of $X$. The information matrix is defined as

$$\ell = P^{-1}$$

(\ell) represents the certainty in the state estimate. If (\ell) is “large” then we have a lot of confidence in our state estimate. In the limit as $\ell \rightarrow \infty$ we have zero knowledge of $X$, and as $\ell \rightarrow 0$ we have perfect knowledge of $X$ [22]. The advantage of using the information filter is that, if the initial uncertainty is infinite, it’s not possible to numerically set $P_0 = \infty$, but it is possible to numerically set $P_0 = 0$. This makes the information filter more mathematically accurate for the zero initial certainty case.
In general, since the filter Kalman and information filter both provide the state estimation, selection of the most appropriate filter depends on the number of states and the number of measurements. But it can be generally said that, the information filter is faster than Kalman filter if \( r > 1.65n \) for time variant systems and \( r > 0.75n \) for time invariant systems (where \( r \) is the number of measurements and \( n \) is the number of states) [26].

The fractional information filter can be implemented as follows:

The measurement update equation for \( P \) can be written as

\[
P_k = \left( \tilde{R}_k \right)^{-1} + H^T R_k^{-1} H
\]

therefore

\[
\left( P_k \right)^{-1} = \left( \tilde{R}_k \right)^{-1} + H^T R_k^{-1} H
\]

This implies that \( (\ell = P^{-1}) \)

\[
\ell_k = \gamma_k + H^T R_k^{-1} H
\]

The time-update equation for \( P \) can be written as:

\[
P_{k+1} = (A(k)+C_j) P_{k-1} (A(k)+C_j)^T + \sum_{j=1}^{n} C_j P_{k-1} C_j^T + Q_{k-1}
\]

\[
\gamma_k = (A(k)+C_j) \ell_k (A(k)+C_j)^T + \sum_{j=1}^{n} C_j \ell_k C_j^T + Q_{k-1}
\]

Therefore, the information filter algorithm for the fractional-order state-space system can be expressed with the following steps

1. The dynamic equations of fractional order state space system is as follows:

\[
\Delta^\alpha X(k+1) = AX(k) + Bu(k) + w(k)
\]  

\[
X(k+1) = \Delta^\alpha X(k+1) - \sum_{j=1}^{n} Y_j (X(k+1-j))
\]  

\[
Y(k) = HX(k) + v(k)
\]  

2. Initialization:

\[
\check{X}(0) = E[X(0)]
\]  

\[
\ell_0 = \left[ E[(X(0) - \check{X}(0))(X(0) - \check{X}(0))^T] \right]^{-1}
\]  

3. For every time step we have

\[
\Delta^\alpha \check{X}(k+1) = A(k) \check{X}(k) + B(k) u(k)
\]  

\[
\check{X}(k+1) = \Delta^\alpha \check{X}(k+1) + \sum_{j=1}^{n} C_j \check{X}(k+1-j)
\]  

\[
\ell_k = \tilde{\gamma}_k + H^T R_k^{-1} H
\]  

\[
K(k) = (\ell_k)^{-1} H^T R_k^{-1}
\]  

\[
\tilde{\gamma}_k = \left[ (A(k)+C_j) \ell_k (A(k)+C_j)^T + \sum_{j=1}^{n} C_j \ell_k C_j^T + Q_{k-1} \right]^{-1}
\]

V. THE CONCURRENT IDENTIFICATION AND STATE ESTIMATION ALGORITHM

In this part of the present paper a unified algorithm for concurrent parameter identification and state estimation is propose. Applying the recursive least square identification algorithm introduced in section III, the system parameters of fractional order state space system are identified and based on which, \( \hat{A} \) and \( \hat{B} \) matrices are properly formed.

Utilizing these matrices in Kalman filter, the state variables of the system are also estimated concurrently. Afterwards, \( \hat{\phi}(k) \) and \( \hat{N}(k) \) are formed using the estimated states and they will be utilized in the next iteration of parameter identification.

\[
J(\theta) = \sum_{j=1}^{n} \left[ y_j(j) - \phi_j(j)(\theta - N(k)) \right]^2
\]

\[
\hat{\theta}_{k+1} = \hat{\theta}_k + F_{k} \dot{\phi}(k) e_{k}(k+1)
\]

\[
F_{k+1} = F_k - \frac{F_k \dot{\phi}(k) \dot{\phi}^T(k) F_k}{1 + \dot{\phi}^T(k) F_k \dot{\phi}(k)}
\]

\[
e_{k+1} = \frac{y_j(j) - \dot{\phi}_j(k) \hat{N}(k)}{1 + \dot{\phi}^T(k) F_k \dot{\phi}(k)}
\]

\[
\hat{\phi}(k) = \left[ \hat{x}_1(k-n) \quad \hat{x}_2(k-n) \quad \cdots \quad \hat{x}_n(k-n) \right]
\]

\[
\hat{u}(k-1) = \left[ u(k-2) \quad u(k-1) \right]^T
\]

\[
\hat{N}(k) = c_{11} \hat{x}_1(k-1) + c_{12} \hat{x}_2(k-2) + \cdots + c_{m-1} \hat{x}_m(k-n)
\]

The identified parameters are used to form \( \hat{A}, \hat{B} \) and \( \hat{\theta} \) as follows:

\[
\hat{\theta} = \left[ \hat{a}_1 \quad \hat{a}_2 \quad \cdots \quad \hat{a}_n \quad \hat{b}_1 \quad \hat{b}_2 \quad \cdots \quad \hat{b}_s \right]^T
\]

\[
\hat{A} = \left[ \begin{array}{cccccc}
0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 \\
\end{array} \right]
\]

\[
\hat{B} = \left[ \begin{array}{cccc}
\hat{b}_1 \\
\hat{b}_2 \\
\vdots \\
\end{array} \right]
\]

In consequence, \( \hat{A} \) and \( \hat{B} \) matrices are used in the fractional information filter as follows:

\[
\Delta^\alpha \check{X}(k+1) = \hat{A}(k) \check{X}(k) + \hat{B}(k) u(k)
\]

\[
\check{X}(k+1) = \Delta^\alpha \check{X}(k+1) + \sum_{j=1}^{n} C_j \check{X}(k+1-j)
\]

\[
\check{X}(k) = X(k) + K(k) (Y(k) - HX(k))
\]

\[
\ell_k = \gamma_k + H^T R_k^{-1} H
\]

\[
K(k) = (\ell_k)^{-1} H^T R_k^{-1}
\]
\[
\hat{\kappa}(k) = \left[ (\hat{A}(k) + C_2 \hat{C}_2 \hat{A}(k) + \hat{C}_2 \hat{C}_2^T + \hat{Q}_c) \right]^{-1}
\]

(49)

The algorithm is implementable following the instructions below:

1) Initialization at time instance \(K=1\), (following the instructions in example)
2) Forming the \(\hat{A}(k)\) and \(\hat{N}(k)\).
3) Calculation of \(F_{k+1}, \varepsilon(k+1)\)
4) Updating the parameter identification vector \(\hat{\theta}_k\)
5) Reading \(a_i, b_i\) parameters from \(\hat{\theta}_k\) vector
6) Forming \(\hat{A}(k), \hat{B}(k)\) matrices
7) Calculating covariance matrix \(\ell_k\) and gain vector \(K(k)\)
8) State vector estimation and calculating \(\hat{X}(k+1)\)
9) Increasing \(k\) and going to 2.

VI. SIMULATION RESULTS

Let us implement and verify the performance of our algorithm in the following exemplary system.

Example:

To verify the accuracy and performance of the proposed system in section V, a 1-input 4-output system with following system matrices is used as an example.

\[
A = \begin{bmatrix}
0 & 1 \\
a_{21} & a_{22}
\end{bmatrix}, B = \begin{bmatrix} b_1 \\
0 \\
0.9 \\
0.8
\end{bmatrix}, H = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0.9 & 0 & 0.8 & 0 \\
0.8 & 0.9 & 0 & 0.7
\end{bmatrix}
\]

where the exact parameters are:
\(a_{21} = -0.4, \ a_{22} = -0.8\)
\(b_1 = -0.2, \ b_2 = -0.6\)
\(y_1 = 0.7, \ y_2 = 0.2\)

In this specific example the elements in Eq. 2 are supposed to be finite, for example \(L\) in this example, which simplifies the calculations. Supposing a rational value for \(L\), the calculation error will be ignorable. As a result, equations in 2 will become as following:

\[
X(k+1) = \Delta^k X(k+1) + \sum_{j=1}^{L} (-1)^{j+1} \begin{bmatrix} y_1 \\ j \end{bmatrix} X(k+1-j)
\]

(50)

The state equations will be like:

\[
\Delta^k X(k+1) = \begin{bmatrix}
0 & 1 \\
-0.4 & -0.8
\end{bmatrix} X(k) + \begin{bmatrix}
-0.2 \\
-0.6
\end{bmatrix} u(k)
\]

\[
X(k+1) = \Delta^k X(k+1) + \sum_{j=1}^{L} \begin{bmatrix}
0.7 \\
0
\end{bmatrix} \begin{bmatrix} 0 \\ j \\
0 \\
0.2 \\
0.7
\end{bmatrix} X(k+1-j)
\]

\[
Y(k) = \begin{bmatrix}
1 & 0 \\
0.9 & 0 \\
0.8 & 0 \\
0.7 & 0
\end{bmatrix} X(k) + v(k)
\]

The simulations are performed using \(L = 50\) to make the approximation errors negligible. Fig I and II consist of identified parameters for \(L = 50\). It is obvious that the parameters are well identified.

![Fig. I. The identified parameters \(a_{21}, a_{22}\) for \(L = 50\).](image1.png)

![Fig. II. The identified parameters \(b_1, b_2\) for \(L = 50\).](image2.png)
VII. CONCLUSION

The recursive identification algorithm presented in this paper is a new approach to estimate the parameters of a fractional order state space system. The basis of the proposed method is the hierarchical identification principle which reduces the computational complexity and run-time of the proposed method. Furthermore, a novel fractional order information filter was provided for state estimation that due to the characteristics of the filter reduces the execution time.

REFERENCES


