Character products for table algebras

Javad Bagherian
Department of Mathematics, University of Isfahan
P.O. Box: 81746-73441, Isfahan, Iran

Amir Rahnamai Barghi
Department of Mathematics, K. N. Toosi University of Technology
P.O. Box: 16315-1618, Tehran, Iran

Abstract

In this paper, we define the character products for table algebras and give a condition in which the products of two characters are characters.

Keyword and phrases: table algebra, character product.

AMS subject Classification 2010: 20C99.

1 Introduction

One of important results in the character theory of finite groups is the Burnside-Brauer Theorem. This theorem states that if a finite group $G$ has a faithful character $\chi$ which takes $k$ values on $G$, then every irreducible character of $G$ is a constituent of one of the characters $\chi^i$ for $0 \leq i < k$.

One of important results in this paper is to state and prove an analog of the Burnside-Brauer Theorem for table algebras. Therefore, we deal with products of characters in table algebras. We mention that products of characters in table algebras need not be characters in general. In order to provide a condition in which the products of characters of a given table algebra are characters, we need to observe the relationship between the characters of a table algebra and the characters of its quotient. Throughout this paper we follow from [1] for the definition of non-commutative table algebras and related notions. Hence we deal with non-commutative table algebras as the following:

A non-commutative table algebra $(A, B)$ is a finite dimensional algebra $A$ over the complex field $\mathbb{C}$ and a distinguished basis $B = \{b_1 = 1_A, \cdots, b_d\}$ for $A$, where $1_A$ is a unit, such that the following properties hold:

(I) The structure constants of $B$ are nonnegative real numbers, i.e., for $a, b \in B$:

$$ab = \sum_{c \in B} \lambda_{abc}c, \quad \lambda_{abc} \in \mathbb{R}^+ \cup \{0\}.$$ 

(II) There is a semilinear involutory anti-automorphism (denoted by $^*$) of $A$ such that $B^* = B$.

(III) For $a, b \in B$ the equality $\lambda_{ab1_A} = \delta_{ab} |a|$ holds where $|a| > 0$ and $\delta$ is the Kronecker symbol.

(IV) The mapping $b \to |b|, b \in B$ is a one-dimensional linear representation of the algebra $A$ such that $|b| = |b^*|$ for all $b \in B$ which is called the degree map.
Let \((A, B)\) be a table algebra. Then \([1, \text{Theorem 3.11}]\) implies that \(A\) is semisimple. The value \(|b|\) is called the degree of the basis element \(b\). From condition (IV) we see that \(|b| = |b^*|\) for all \(b \in B\). Therefore, for an arbitrary element \(\sum_{b \in B} x_b b \in A\), we have \(|\sum_{b \in B} x_b b| = |\sum_{b \in B} x_b b|\).

For each \(a = \sum_{b \in B} x_b b\), we set \(a^* = \sum_{b \in B} \overline{\pi_b} b^*\), where \(\overline{\pi}\) means the complex conjugate of \(\pi_b\). For any \(x = \sum_{b \in B} x_b b \in A\), denote by \(\text{Supp}(x)\) as the set of all basis elements \(b \in B\) such that \(x_b \neq 0\). If \(E, D \subseteq B\), then we set \(ED = \bigcup_{e \in E, d \in D} \text{Supp}(ed)\).

A nonempty subset \(C \subseteq B\) is called a closed subset, if \(C^* C \subseteq C\). We denote by \(\mathcal{C}(B)\) the set of all closed subsets of \(B\). In addition, \(C \in \mathcal{C}(B)\) is said to be normal in \(B\) if \(bC =Cb\) for every \(b \in B\), and denote it by \(C \trianglelefteq B\). A closed subset \(C\) of \(B\) is said to be strongly normal and denoted by \(C \triangleleft B\), if for each \(b \in B\)

\(b' C b \subseteq C\).

## 2 Main Results

**Theorem 2.1** Let \(C \trianglelefteq B\). Then the map \(\pi : A \rightarrow A/C\) defined by \(\pi(b) = \frac{|b|}{|b/C|} (b/C)\) is an algebra homomorphism.

**Theorem 2.2** Let \(C \trianglelefteq B\) and let \(\psi : A/C \rightarrow \text{Mat}_n(\mathbb{C})\) be a representation of \(A/C\). Then \(\overline{\psi} : A \rightarrow \text{Mat}_n(\mathbb{C})\) defined by \(\overline{\psi}(b) = \frac{|b|}{|b/C|} \psi(b/C)\) is a representation of \(A\).

For an associative algebra \(A\), the tensor product \(V \otimes W\) of two \(A\)-modules \(V\) and \(W\) is a vector space, but not necessarily an \(A\)-module. In order to make an \(A\)-module on \(V \otimes W\), there must be a linear binary operation \(\Delta : A \rightarrow A \otimes A\) which is also an algebra homomorphism. This is an important property for the algebra \(A\) becomes a Hopf algebra. For instance, in group theory the tensor products of two \(G\)-modules \(V\) and \(W\) gives us a module, indeed the group algebra \(\mathbb{C}G\) is a Hopf algebra with \(\Delta : g \rightarrow g \otimes g\). So if \(\chi\) and \(\psi\) are afforded by two \(G\)-modules, then their tensor product affords the character \(\chi \psi(g) := \chi(g) \psi(g)\) which is called the character product of \(\chi\) and \(\psi\).

In general, a table algebra \((A, B)\) is not a Hopf algebra and so it is not generally possible to define the structure of an \(A\)-module on \(V \otimes W\). In \([2]\) Doi introduced a generalization of Hopf algebras and defined a binary linear operation \(\Delta : b \rightarrow \frac{1}{|b|} b \otimes b\), \(b \in B\). By considering this binary linear operation, we define the character product of \(\chi\) and \(\psi\) by:

\[
\chi \psi(b) := \frac{1}{|b|} \chi(b) \psi(b), \quad b \in B. \tag{1}
\]

Let \(V\) and \(W\) be \(A/C\)-module and \(A\)-module, respectively. We define a multiplication of \(A\) on \(V \otimes W\) as the following:

\[
b(v \otimes w) := (b/C)v \otimes bw, \quad v \in V, \ w \in W, \ b \in B. \tag{2}
\]

In the blow we assume that \((A, B)\) is a table algebra with a strongly normal closed subset \(C\) and \(e = |C|^{-1} C^+\).

**Theorem 2.3** Let \(V\) be an irreducible \(A\)-module with \(\dim_{\mathbb{C}}(e V) \neq 0\) and let \(W\) be an \(A\)-module. Then \(V \otimes W\) is an \(A\)-module given by the multiplication in (2).

**Theorem 2.4** Let \(\chi\) be an irreducible character of \(A\) such that \(\chi(e) \neq 0\). Then \(\chi \psi\) is a character of \(A\), where \(\psi\) is a character of \(A\).

Let \(\chi\) be a character of table algebra \((A, B)\). The following subset of \(B\)

\[
K(\chi) = \{b \in B : \chi(b) = |b| \chi(1)\}
\]
is a closed subset of $B$ (see [3]).

Below we give our main result which proves the Burnside-Brauer Theorem on finite groups for table algebras.

**Theorem 2.5** Let $(A, B)$ be a table algebra. Suppose that $A$ has a character $\chi$ with $K(\chi) = \{1_A\}$ such that $\chi(b)/|b|$ takes on exactly $k$ different values for $b \in B$. If all powers of $\chi$ by itself is a character, then each irreducible character of $A$ appears as an irreducible component of one of $\chi^i$, where $0 \leq i \leq k - 1$ and $\chi^0 = \rho$.

**References**

