Robust Optimal Control using Sliding-Mode Design: Unit Vector Approach

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Abstract: In this paper, two robust optimal control methods based on sliding mode design are introduced and implemented using the well-known unit vector approach. Time-varying sliding surfaces are implemented to solve linear optimal problems. The introduced sliding surfaces produce optimal controllers with respect to quadratic cost function. The presented design procedure is much easier than solving an HJB equation if one needs to achieve smaller costs than LQR controlled system (in which the controller designed for the linearized model and implemented to the original nonlinear model). Domain of attraction of closed-loop system is also bigger than LQR controlled system. The introduced methods are applied to an inverted pendulum system and the simulation results are presented and compared.

Keywords: sliding mode, unit vector approach, inverted pendulum, linear optimal control, and quadratic cost minimization.

1. Introduction

Variable structure control (VSC) has been widely recognized as a powerful control strategy due to its ability to make a control system very robust. Sliding mode control is a particular type of variable structure control systems which is designed to drive and constrain the system to lie within a neighborhood of the sliding surface [1,4,5]. There are two main advantages of this approach. Firstly, the dynamic behavior of the system may be tailored by the particular choice of sliding surface. Secondly, the closed-loop response becomes totally insensitive to a particular class of uncertainties and disturbances. This design approach comprises two components: the design of a surface in the state space so that the sliding motion satisfies the specifications imposed by designer; and the synthesis of a control law, discontinuous about the sliding surface, such that the trajectories of the closed-loop system are directed toward the surface.

Given the linear uncertain system:

\[ \dot{x} = Ax(t) + Bu(t) + f(t,x,u) \]  \hspace{1cm} (1)

where \( f(t,x,u) \) is assumed to be unknown but bounded by some known functions of the state. A sliding surface constructed with a constant matrix \( S \):

\[ Sx = 0 \]  \hspace{1cm} (2)

has been shown to play a crucial role in providing insensitivity to the uncertainties and disturbance rejection if the close-loop feedback trajectory remains on this surface. More precisely, given the system (1), there exists a linear surface (2) parameterized by the constant matrix \( S \) such that the motion on this surface is asymptotically stable, i.e. the system, whose dynamics completely characterize this sliding mode, has its eigenvalues in the open left-half complex plane. The design of the sliding manifold by minimizing a cost function in which the integrand is quadratic in term of the state is proposed in [7]. The method will enable desirable weightings to be placed upon particular elements [1]. This method is implemented to shape the time response of the closed loop system. In [2] a class of time-varying sliding-mode manifolds for solving robust linear optimal control problems is considered. Two special cases of these surfaces are used to provide robustness to linear
optimal regulators. The design approach can be applied to nonlinear plant dynamics when the plant is linearized about the equilibrium point. Using these methods, domain of attraction is guaranteed to be enlarged.

To study the above methods, the inverted pendulum on a cart is considered. The method in [1] is used to gain a controller with the desired time response properties.

The control law in the sliding mode approach usually involves discontinuous components or their approximations, which bring difficulties in mathematical analysis. In this paper, approximations of two different discontinuous components are used with the sliding-surfaces introduced in [2]: tanh and approximation of unit vector approach. Simulation results are compared with the LQR method. Based on the simulations, new results in optimal sliding-mode are obtained. This paper is organized as follows: In Section 2 the first idea in optimal sliding mode using a time-invariant sliding surface is considered. Section 3 demonstrates sliding mode controller design for optimal quadratic regulators. Section 4 presents the unit vector approach to use with the design method of section 3. In Section 5 the inverted pendulum model, used for this study is presented. Simulation results are presented in Section 6. Finally, in Section 7 several conclusions are made.

2. Optimization and Sliding Mode (using Time Invariant Sliding Surface)

The first idea in optimal sliding mode is formulated by Utkin and Young [7] in which a sliding mode surface is determined by optimizing the following quadratic cost (on the sliding surface):

\[ J = \frac{1}{2} \int_{t_i}^{\infty} x^T Q x \, dt \]  

whose \( Q \) is both symmetric and positive definite and \( t_i \) is the time at which the sliding motion commences. The aim is to minimize equation (3) subject to the system equation:

\[ \dot{x}(t) = Ax(t) + Bu(t) \]

\[ x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \]

A sliding surface of the form (2) can be used to solve the problem. Here the detailed discussions will be eliminated, but it is important to note that in general case, the system is not initially on the sliding manifold and the designer must enforce the states to reach the sliding manifold in finite time.

Only on the sliding manifold, (3) will be minimized. In the next section, other methods to minimize another quadratic cost will be presented.

3. Sliding Mode Controller Design for Optimal Quadratic Regulators (using Time Varying Sliding Surface)

In this section, two different sliding surface designs to solve optimal sliding mode problem is discussed in which the following quadratic cost function will be minimized:

\[ J = \frac{1}{2} \int_0^\infty x^T Q x + u^T u \, dt \]  

The following theorem summarizes a sliding mode manifold design using co-states for linear-quadratic optimal regulators [2].

**Theorem 1** Let the m-dimensional sliding mode manifold be defined by

\[ S = S_0 B x + S_0 A x \]  

where \( S_0 B \) is nonsingular, \( \det[S_0 B] \neq 0 \) and \( P \) is the positive definite solution of the algebraic Riccati equation:

\[ P A + A^T P - P B B^T P + Q = 0 \]  

If \( S_0 x(t_0) = 0 \) then the motion of the system (4) and (7) satisfies the equation of the manifold (6) for all \( t \geq t_0 \) (method 1).

If we choose to pick the feedback gains in the co-state equation for sliding mode design to be the linear optimal regulator feedback gains, the dynamics on the sliding surface, are identical to the linear-quadratic optimal feedback system:

\[ \dot{x} = (A - B B^T) x \]

then (5) will be minimized:

\[ J = \frac{1}{2} \int_0^\infty x^T Q x + u^T u \, dt \]

From theorem 1 it is considered that by choosing the sliding surface as (6) and (7) and satisfying the following conditions linear optimal problem can be solved through sliding mode design:

\[ \det[S_0 B] \neq 0 \]

\[ S_0 x(t_0) = 0 \]

The first optimal sliding manifold design that provides a robust feedback control implementation (theorem 1) is characterized by a family of functionals that utilize the inner product
of the state and co-state vectors. The second sliding mode design approach is introduced in theorem 2 [2]. Here, co-states play an essential role in the construction of optimal sliding surfaces.

**Theorem 2** an optimal sliding surface for the linear optimal regulator is an m-dimensional manifold of the form

\[ s = [s_1, s_2, \ldots, s_m] \]

\[ s_i = \frac{1}{2} [x(t)^T P x(t) - \omega(t)] \]

\[ \dot{s}_i = S_i(t)x(t) \quad i = 2, \ldots, m \]

where

\[ \dot{\omega} = x^T Q x \]

\[ \omega(t_0) = x_0 P x_0 \]

\[ S_i(t_0) = 0 \]

and \( P \) is the positive definite solution of the algebraic Riccati equation (8):

\[ PA + A^T P - PBB^T P + Q = 0 \]

then the quadratic cost function (5) will be minimized:

\[ J = \frac{1}{2} \int_0^\infty xQx + u^T u \, dt \]

The above approach can also be used to solve finite horizon problems (method 2).

4. **Unit Vector Approach**

The control law \( u(t) \) to induce the sliding motion is given by

\[ u(t) = u_{eq}(t) + u_n(t) \]

where \( u_{eq}(t) \) is the equivalent control, which is obtained from \( \dot{s} = 0 \). On the other hand, the nonlinear component \( u_n(t) \) can be defined in several forms as follows:

1) \[ u_n(t) = -\rho(t,x) \text{sgn}(s) \]

2) \[ u_n(t) = \begin{cases} -\rho(t,x) \frac{s}{|s|} & (s \neq 0) \\ 0 & (s = 0) \end{cases} \]

The control structure obtained from (16) is often referred to as a scaled relay structure. An alternative choice is (17) which is referred to as unit vector approach and can be used in multivariable systems. Here to reduce chattering, continuous approximation of the above discontinuous control laws are used:

\[ 1) \, u_n(t) = -\rho(t,x) \tanh(s) \]  
\[ 2) \, u_n(t) = \begin{cases} -\rho(t,x) \frac{s}{|s|} & (s \neq 0) \\ 0 & (s = 0) \end{cases} \]

where in (19) \( \delta \) is a small positive constant.

5. **Inverted Pendulum Model**

In many control exercises the inverted pendulum system is used as a highly nonlinear benchmark to experiment the advantages of control approaches [1,6]. Consider the inverted pendulum with a cart shown in Figure 1. The pivot of the pendulum is mounted on a cart which can move in a horizontal direction. The cart is driven by a motor that exerts a horizontal force \( f(t) \) on the cart. This force is the only control input to the system. By manipulating \( f(t) \), we would like to control the position of the pendulum. From the physical description of the system, it is clear that balancing the pendulum in a vertical position is open-loop unstable. The cart is to be manipulated so that the pendulum remains in an upright position by the controller. The equations of the motion are:

\[ \begin{align*}
\dot{\theta} &= \frac{F}{m} - \frac{ml}{M+m} \sin \theta - \frac{M}{M+m} \cos \theta \\
\dot{\dot{x}} &= \frac{F}{m} \\
J\dot{\theta} + F_\theta \dot{\theta} - m g \sin \theta &= 0
\end{align*} \]

with \( u=f(t) \) as control input. Solving these equations for \( \dot{\theta} \) and \( \ddot{x} \), and using the following state variables:

\[ \begin{align*}
x_1 &= x & \text{Cart position (m)} \\
x_2 &= \theta & \text{Link angle (rad)} \\
x_3 &= \dot{x} & \text{Cart velocity (m/sec)} \\
x_4 &= \dot{\theta} & \text{Link angular velocity (rad/sec)}
\end{align*} \]

yields the state equations:

\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{F}{m} - \frac{ml}{M+m} \sin x_2 - \frac{M}{M+m} \cos x_2 \\
\dot{x}_3 &= \frac{F}{m} \\
\dot{x}_4 &= \frac{F_\theta}{J} - \frac{m g}{J} \sin x_2
\end{align*} \]
\[ \dot{x}_1 = x_3 \]
\[ \dot{x}_2 = x_4 \]
\[ \dot{x}_3 = \frac{1}{(M + m)J - m^2l^2 \cos^2(x_2)} (-F_x J x_3 + mlJx_2 \sin x_2 + F_0 mlx_4 \cos x_2 - m^2l^2 \sin x_2 \cos x_2 + Ju) \]
\[ \dot{x}_4 = \frac{1}{(M + m)J - m^2l^2 \cos^2(x_2)} (F_x mlx_3 \cos x_2 - m^2l^2 \cos x_2 \sin x_2 - (M + m)F_0 x_4 + (M + m)mg \sin x_2 - ml \cos(x_2) u) \]  
(21)

The values of the physical parameters used are given in table 1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>M (kg)</td>
<td>3.2</td>
</tr>
<tr>
<td>m (kg)</td>
<td>0.535</td>
</tr>
<tr>
<td>J (kg m²)</td>
<td>0.062</td>
</tr>
<tr>
<td>l (m)</td>
<td>0.365</td>
</tr>
</tbody>
</table>

Table 1. Physical parameters

A linearization of the system (21) has been made about the equilibrium point at the origin and results in:

\[
A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1.9333 & -1.9872 & 0.0091 \\ 0 & 369771 & 62589 & -0.1738 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0.3205 \\ -1.0095 \end{bmatrix} \]  
(22)

In next section, controllers are designed based on the model (22) but they’re applied to the nonlinear model (21).

6. Simulation results

Simulations presented here, are the results of the proposed methods on inverted pendulum system in section 5. In the first experiment, the method discussed in section 2 is applied to the system to consider that it can be used to shape the time response. Simulation results are shown in Figure 2a,b. In 2a the weights of all the states are equal and the system settles at origin with the same speed for all states. But in 2b the weight of \( \theta \) is the largest and other weights are equal. Therefore \( \theta \) reaches to zero more rapidly.

In Fig. 3a,b the performance of the system that is controlled by an LQR controller is shown. In 3a the initial condition is \( x_0 = [0 \ 1.19 \ 0 \ 0]^T \) and the system is stable. But when \( \theta_0 \) changes to 1.2(Rad) in 3b, the system will became unstable.

In the remainder of this section, Figures are related to section 3 and addressed as follows:

a: method 1 used with approximated unit vector approach
b: method 1 implemented with \( \tanh \) function
c: method 2 used with \( \tanh \) function (results of method 2 with unit vector approach is similar to \( \tanh \) function).

For the optimal sliding mode controlled inverted pendulum, closed loop system remains stable with the initial condition: \( x_0 = [0 \ 1.2 \ 0 \ 0]^T \) (Figure 4a,b,c).

In order to investigate the effects of uncertainty on performance of the controllers we perform other simulations. 5a,b,c are the simulation results when \( x_2 \) is changed to \( 0.1 \sin(x_1) \) and initial states are the same as 3a in which LQR controlled system was unstable. From the figures, it is obviously understood that method 1 saves the time response of figure 3a.

Figures 6, 7 and 8 show the cost of different methods in the mentioned conditions. It can be concluded that method 1 can be implemented to reduce the cost of LQR controller applied to original nonlinear model. It is important to note that method 1 used with approximated unit vector approach, yields the smallest cost.

7. Conclusion

In this paper some procedures for designing optimal sliding mode controller are presented and applied to an inverted pendulum system. The performances of the controller with exact parameter information and in presence of uncertainty are discussed. The simulations show a good robustness of the optimal sliding mode control methods to achieve optimal controllers in comparison with LQR controller, designed for linearized model and applied to nonlinear model. Two methods are introduced to achieve robust optimal controller. Simulation results show that method 1 can be implemented to reduce the cost of LQR controller applied to original nonlinear method. The minimum cost is obtained from method 1 used with approximated unit vector approach. Also it is obviously understood that method 1 saves the time response in the presence of uncertainties. Then it can be used to design suboptimal controller when solving an HJB equation is difficult or impossible.
Acknowledgements

The financial support of control and intelligent processing center of excellence of Tehran University is acknowledged.

References

Figure 4c: Time response of the optimal sliding mode controlled system, method 2 and unit vector, $x_0 = [0 \ 1.20 \ 0 \ 0]^T$

Figure 5a: System with the uncertainty, method 1 and unit vector, $x_0 = [0 \ 1.19 \ 0 \ 0]^T$

Figure 5b: System with the uncertainty, method 1 and $tanh$, $x_0 = [0 \ 1.19 \ 0 \ 0]^T$

Figure 5c: System with the uncertainty, method 2 and unit vector, $x_0 = [0 \ 1.19 \ 0 \ 0]^T$

Figure 6: Costs of the LQR and optimal sliding mode controlled system, $x_0 = [0 \ 1.19 \ 0 \ 0]^T$

Figure 7: Costs of the optimal sliding mode controlled system, $x_0 = [0 \ 1.20 \ 0 \ 0]^T$

Figure 8: Costs of the optimal sliding mode controlled system with uncertainty, $x_0 = [0 \ 1.19 \ 0 \ 0]^T$