Abstract: In this paper, stability analysis of the PDC-based Takagi-Sugeno (T-S) fuzzy impulsive control systems by using the extended Lyapunov stability theory is investigated. Some sufficient conditions and criterions are derived for stability of T-S fuzzy impulsive control system in terms of linear matrix inequalities (LMIs). Finally, control schemes are successfully applied to stabilize a predator-prey system. The simulations demonstrate the effectiveness and advantages of the proposed approach.

Keywords: Fuzzy impulsive control, Takagi–Sugeno (T–S) fuzzy model, Lotka-Volterra, Predator-prey System.

1. Introduction

Volterra presented the differential equation to solve the issue of the sharp change of the population of the sharks (predator) and the minions (prey) in 1925. After that, the predator-prey system has been applied to many areas and played an important role in the biomathematics. Much attention has been attracted to the stability of the predator-prey system. F. Brauer and A. C. Soudack studied the global behavior of a predator-prey system under constant rate prey harvesting with a pair of non-linear ordinary differential equations in [1]. Jian Xu and his workmates concluded that a short-time delay could ensure the stability of the predator-prey system [2]. After analyzing the different capability between the mature and immature predator, Wendi Wang and his workmates obtained the global stability with the small time delay system [3]. Dong Li and his partners studied the impulsive control of Lotka–Volterra predator–prey system and established sufficient conditions of the asymptotic stability with the method of Lyapunov functions [4]. Shengqiang Liu and Jianhua Zhang studied the coexistence and stability of predator-prey model with Beddington- DeAngelis functional response and stage structure [5]. Yanqiu Li did some work on the Holling II functional response and concluded that the positive solution of the associated stochastic delay differential equation [6]. Furthermore, Wonlyul Ko and Kimun Ryu studied the qualitative behavior of non-constant positive solutions on a general Gauss-type predator-prey model with constant diffusion rates under homogenous Neumann boundary condition [7]. Additionally, many papers discussed the predator-prey system with other different methods, such as LaSalle’s invariance principle method [8], impulsive perturbations method [9], and generalized Gauss-type predator–prey model [10], etc.

The Takagi–Sugeno (T–S) fuzzy model is a kind of fuzzy system proposed by Takagi and Sugeno [11], which is described by a set of fuzzy IF–THEN rules to represent local linear input–output relations of a nonlinear system. The main idea of the T–S fuzzy model is to express the local dynamics of each fuzzy rule by a linear system model and to express the overall system by fuzzy “blending” of the local linear system models. This model can utilize the well-established linear system theory to analyze and synthesize highly nonlinear dynamic systems [1]–[3], [12]-[13].

In recent years, fuzzy impulsive theory has been applied to the stability analysis of the non-linear differential equations [14]-[18]. However, it should be admitted that the stability of Fuzzy Logic Controller (FLC) is still an open problem. It is well-known that the parallel distributed compensation technique has been the most popular controller design approach and belongs to a continuous input control way. It is important to point out that there exist many systems, like the predator-prey system, which cannot commonly endure continuous control inputs, or they have impulsive dynamical behavior due to abrupt jumps at certain instants during the evolving processes. In this sense, it is the same with communication networks, biological population management, and chemical control, etc [19]-[25]. Hence, it is necessary to extend FLC and reflect these impulsive jump phenomena in the predator-prey system. Until recently, few papers talks about the stability of Lotka-Volterra predator-prey system with fuzzy impulsive control, for example [26], in which a linear fuzzy impulsive controller is used.

In this paper, the writers will study the stability problem of the predator-prey system by the combination of nonlinear fuzzy impulsive control and PDC-based state
feedback control. The proposed method is used to obtain the controller coefficients and stability analysis which is quite innovative, efficient and more practical than the methods in other papers [26]. In addition, the proposed nonlinear fuzzy impulsive controller is definitely more efficient than the linear one. In this paper, to achieve a better performance than the regular fuzzy impulsive controller the authors have combined a PDC-based state feedback controller with a nonlinear fuzzy impulsive controller which leads to a more flexible and elegant controller.

The rest of this paper is organized as follows. Section 2 describes the T-S fuzzy impulsive system by blending local linear impulsive systems. In Section 3, the theoretic analysis and design algorithm on stability of the fuzzy impulsive system are performed. Numerical simulations for the predator-prey system with impulsive effects are carried out with respect to the proposed methods in Section 4. Finally, some conclusions are made in Section 5.

2. Problem Formulation

In this section, by using the methods introduced in [27], we can construct a Takagi-Sugeno (T-S) fuzzy model as follows:

Rule i: IF $z_1(t)$ is $M_{i1}$, $z_2(t)$ is $M_{i2}$, ..., and $z_p(t)$ is $M_{ip}$, THEN $\dot{x}(t) = A_i x(t) + B_i u(t)$, $(i = 1, 2, ..., r)$

where $r$ is the number of T–S fuzzy rules, $z_i(t)$ is the premise variables, each $M_{ij}(j = 1, 2, ..., p)$ is fuzzy set, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is control input and, $(A_i, B_i)$ is controllable pair of system matrices in which $A_i \in \mathbb{R}^{n \times n}, B_i \in \mathbb{R}^{n \times m}$.

By using the singleton fuzzifier, product inference and the center average defuzzifier, the final output of the fuzzy systems can be represented as:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t)) [(A_i x(t) + B_i u(t))$$

where $h_i(z(t)) = \omega_i(z(t)) \sum_{i=1}^{n} \omega_i(z(t)), \omega_i(z(t)) = \prod_{i=1}^{n} M_{im}(z_{im}(t)), \omega_i(z(t)) \geq 0, \sum_{i=1}^{r} h_i(z(t)) = 1, h_i(z(t)) \geq 0, i = 1, 2, ..., r$. Here, $u(t) = u_1(t) + u_2(t)$ in which $u_1(t)$ and $u_2(t)$ are a PDC-based state feedback control input and a fuzzy impulsive controller, respectively. An impulsive control law of (1) is given by a sequence $\{t_{ik}, I_{jk}(x(t))\}$ in which $0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots, \lim_{k \to \infty} t_k = \infty$, and $I_{jk}(x(t))$ is a continuous function which maps $\mathbb{R}^n$ to $\mathbb{R}^m$ for all $k = 1, 2, ..., 3, \cdots$. Thus, it is named “impulsive function”. In the case of PDC based T-S fuzzy impulsive control systems, the fuzzy impulsive controller shares the same fuzzy sets and rules with T-S fuzzy model but in different time. Consequently, the fuzzy impulsive controller law can be expressed as

$$u_2(t) = \sum_{j=1}^{r} h_j(z(t)) \left( \sum_{k=1}^{\infty} \delta(t - t_k) (I_{jk}(x(t))) \right)$$

By substituting $u_2(t)$, the following formulation of the closed-loop model is obtained

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t)) \left( A_i x(t) + B_i u_1(t) + B_i \sum_{j=1}^{r} h_j(z(t)) \left( \sum_{k=1}^{\infty} \delta(t - t_k) (I_{jk}(x(t))) \right) \right)$$

However, (3) implies that

$$x(t_k + q) - x(t_k - q) = \int_{t_k - q}^{t_k + q} \left[ \sum_{i=1}^{r} h_i(z(t)) \left( A_i x(t) + B_i u_1(t) + B_i \sum_{j=1}^{r} h_j(z(t)) \left( \sum_{k=1}^{\infty} \delta(t - t_k) (I_{jk}(x(t))) \right) \right) \right] dt$$

where $q > 0$ is sufficiently small. As $q \to 0^+$, one obtains

$$x(t_k^+) - x(t_k^-) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z(t)) h_j(z(t)) (B_i I_{jk}(x(t))), k = 1, 2, 3, ...$$

For abbreviation, arguments $t$ and $z(t)$ will be removed from some terms like $x(t)$, $u(t)$ and $h_i(z(t))$. Consequently, they will be shown as $x$, $u$ and $h_i$.

Therefore, the PDC based T-S fuzzy impulsive control system is given as follows:

$$\begin{align*}
\dot{x} &= \sum_{i=1}^{r} h_i(A_i x + B_i u_1) \\
\Delta x|_{t=t_k} &= I_{jk}(x) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j B_i I_{jk}(x), t = t_k \\
\dot{x}(t_k^+) &= x_0 \quad k = 1, 2, 3, ...
\end{align*}$$

where $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-) = I_{jk}(x)$ denotes an impulsive plant in (6).

Some assumption, lemmas and remarks that are required for stability analysis, are presented here.

Remark 1: According to above discussions, it is important to remember that in this paper, the following definitions of the impulsive function $I_{jk}(x)$ are considered.

- **linear case:** $I_{jk}(x) = D_{jk} x$

- **Nonlinear case:** $I_{jk}(x) = D_{jk} x + g_{ij}(t, x)$

where $D_{jk}$s is matrices with appropriate dimension and $g_{ij}(t, x)$ is a nonlinear function satisfying certain conditions stated below.

Assumption 1: It is assumed that the nonlinear function $g_{ij}(t, x)$ satisfies

$g_{ij}^T(t, x) Q g_{ij}(t, x) \leq x^T(t) Q x(t), \forall t \geq 0$

in which $Q$ is a positive-definite matrix.
Remark 2: Assumption 1 is referred to as Lipschitz condition. It is easy to see that Assumption 1 implies that the nonlinear function \( g(t,x(t)) \) satisfies a Lipschitz condition which is usually assumed in literature. Also, Assumption 1 guarantees that \( g_i(t,0) = 0 \) which implies the origin \( (x(t) = 0) \) is the trivial solution of (6).

Lemma 1: ([28]) For any real matrices \( X_i, Y_i \), \( 1 \leq i \leq r \), and \( P > 0 \) with appropriate dimensions, we have
\[
2 \sum_{i=1}^{r} \sum_{j=1}^{r} h_{ij} X_i^T P Y_j \leq \sum_{i=1}^{r} h_i (X_i^T P X_i + Y_i^T P Y_i),
\]
where \( h_i, (1 \leq i \leq r) \) is defined as \( h_i \geq 0, \sum_{i=1}^{r} h_i = 1 \).

Remark 3: It should be noted, that since impulsive term is nonlinear and time-varying, system (6) is time-varying nonlinear impulsive control system.

3. Main Results

In this section, we consider the general conditions for stability analysis of (6) based on Lyapunov method. First, the issues which concern the necessity of impulsive control strategy [29] are addressed and then the main results are given. Consider the following impulsive functional equation
\[
\dot{x}(t) = f(t,x(t)), \quad t \geq t_0
\]
\[
\begin{cases}
\Delta x(t_k) = \xi_k (x(t_k^-), t_k), \\
k \in \mathbb{Z}^+
\end{cases}
\]
where \( f: \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n \) and \( l: [t_k, \infty) \times \mathbb{R}^m \to \mathbb{R}^m \). Then, \( S(\rho) = \{ x(t) \in \mathbb{R}^n | ||x|| < \rho \} \) denotes the space of piecewise right-continuous function. \( ||\cdot|| \) is a norm in \( \mathbb{R}^n \) and \( Z^+ = \{ 1, 2, 3, \ldots \} \). For stability analysis, it is assumed that \( f(t,0) = 0 \) and \( l(t_k,0) = 0 \) so that \( x(t) = 0 \) is trivial solution of (9).

Theorem 1: ([29]) (Liu et al) Assume that:
1) There exist \( \xi_k \in \mathbb{R} \) and \( C_k \in \mathbb{K} \) such that
\[
D^+ V(t,x(t)) \leq \frac{\xi_k}{\Delta t_k} C_k \left( V(t,x(t)) \right), \\
(t,x(t)) \in (t_{k-1}, t_k) \times S(\rho)
\]
where \( K \) is denoted as the class of continuous functions. \( \sum \) the class of function \( V(t,x(t)) \): \( R_+ \times R^n \to R_+ \) such that \( V(t,x(t)) \) is a positive definite, locally Lipschitzian in \( x(t) \), continuous everywhere except possibly at a sequence of points \( \{ t_k \} \) at which \( V(t,x(t)) \) is left continuous and the right limit \( V(t_k^+,x(t_k^+)) \) exists for all \( x(t) \in \mathbb{R}^n \). It is also assumed that, there exists a \( \rho_1 \in (0, \rho) \) such that \( x(t) \in S(\rho_1) \) implies \( x(t) + l(t_k x(t)) \in S(\rho) \).
2) There exists a \( v_k \in \mathbb{R} \) and \( d_k \in \mathbb{K} \) such that
\[
V(t_k^+,x(t_k^+)) + v_k d_k (V(t_k^+,x(t_k^+)), x(t) \in S(\rho)
\]
\[
3) \xi_k + v_k \leq 0, \text{ for } s \in (0, \rho), c_k(s) \leq d_k(s) \text{ if } v_k < 0 \text{ and } d_k(s) \leq c_k(s) \text{ if } v_k < 0
\]
Then, impulsive control system (9) is stable.

Now, we are ready to present stability theorems for T-S fuzzy impulsive system (6). First, the PDC-based fuzzy impulsive control system (6) with Non-linear controller \( (I_{jk}(x) = D_{jk} x + g_j(t,x)) \) is considered. Theorem 2 gives sufficient conditions to stabilize (6) in which \( I_{jk}(x) = D_{jk} x + g_j(t,x) \) and \( u_i(t) = \sum h_i F_i x(t) \) are nonlinear impulsive function and PDC-based state feedback controller, respectively. (\( F_i \)'s is gain of state feedback controller)

Theorem 2: The origin of the PDC-based T-S fuzzy impulsive control system (6) is stable if there exist a positive-definite matrix \( P \), scalars \( \xi_k 's \), \( v_k 's \), positive scalars \( \Delta t_k 's \), \( \varepsilon 's \) and matrices \( D_{jk} 's \) such that the following conditions hold:
\[
\begin{align}
\xi_k + v_k &\leq 0, & v_k \geq -1 \\
T &\geq 0 & T &\geq 0 & T &\geq 0 \quad (10a) \\
\tilde{V}_{il_k} &< 0 & \tilde{V}_{il_k} &< 0 & \tilde{V}_{il_k} &< 0 \quad (10b) \\
T &\geq 0 & T &\geq 0 & T &\geq 0 \\
(1 + \varepsilon _1 + \varepsilon _2) \lambda_{imin} &< 0 \\
(10c)
\end{align}
\]
where \( \tilde{V}_{il_k} = T A_i^T + A_i T + 2 TF_i B_i^T - \frac{\xi_k}{\Delta t_k} T, \lambda_{imin} \) is the smallest eigenvalue of \( B_i^T P B_i \), \( \tilde{V}_{il_k} = B_i D_{jk} T + T D_{jk} B_i^T - \varepsilon_1 \), \( T = P^{-1} \), \( V_{il_k} = B_i D_{jk} T (i,j) \) and \( r \) is number of fuzzy rules.

Proof: Consider the Lyapunov function \( V(t,x) = x^T(t) P x(t), x(t) \in (t_k^-, t_k) \) for the PDC-based T-S fuzzy impulsive control system (6) with nonlinear controller \( I_{jk}(x) \) and PDC-based state feedback controller \( u_i(t) \). Now, from condition 1 of Theorem 1 and by taking the upper right-hand generalized derivative of the Lyapunov function along the trajectories of system (6), the following is obtained
\[
D^+ V(t,x(t)) = \frac{\xi_k}{\Delta t_k} x^T(t) P x(t)
\]
\[
= \frac{\xi_k}{\Delta t_k} x^T(t) P x(t) + x^T(t) P \dot{x}(t) - \frac{\xi_k}{\Delta t_k} x^T(t) P x(t)
\]
\[
= \sum_{i=1}^{r} h_i \left( A_i x(t) + \sum_{j=1}^{r} h_{ij} B_j F_i x(t) \right) P x(t)
\]
\[
+ x^T(t) P \sum_{j=1}^{r} h_j B_j F_i x(t)
\]
\[
- \frac{\xi_k}{\Delta t_k} x^T(t) P x(t)
\]
\[
= \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j x^T(t) \left( A_i^T P + P A_i + 2 F_i^T B_i^T P \right)
\]
\[
- \frac{\xi_k}{\Delta t_k} P x(t) < 0 \quad (11)
\]
If \( \Gamma_{il_k} = A_i^T P + P A_i + 2 F_i^T B_i^T P - \frac{\xi_k}{\Delta t_k} P < 0 \), then
\[
D^+ V(t,x) - \frac{\xi_k}{\Delta t_k} x^T(t) P x(t) < 0. \text{ By pre- and post- multiplying (10) by } T = P^{-1}, \text{ we can get}
\]
\[
\tilde{V}_{il_k} = T A_i^T + A_i T + 2 TF_i B_i^T - \frac{\xi_k}{\Delta t_k} T \quad (12)
\]
From (8), if $TA_i^T + A_iT + 2TF_iB_i^T - \frac{k_i}{\Delta k} T < 0$ then

$$D^*V(t,x) - \frac{k_i}{\Delta k} x^T P x < 0.$$  

From condition 2 of Theorem 1 we get:

$$V(t^*_k, x) - V(t_k, x) - v_k d_k(V(t_k, x)) = x^T(t) P I_k(x(t)) + I_k^T(x(t)) Px(t) + I_k^T(x(t)) P I_k(x(t)) - x^T(t) P x(t) - v_k x^T(t) P x(t)$$

$$= x^T(t) P \left( \sum_{i=1}^r h_i h_j B_i L_{jk} (x(t)) \right) + \left( \sum_{i=1}^r \sum_{j=1}^r h_i h_j B_i L_{jk} (x(t)) \right)^T P x(t) + \left( \sum_{i=1}^r \sum_{j=1}^r h_i h_j B_i L_{jk} (x(t)) \right) P \left( \sum_{i=1}^r \sum_{j=1}^r h_i h_j B_i L_{jk} (x(t)) \right)$$

$$= x^T(t) \left( \sum_{i=1}^r h_i h_j B_i \left( D_{jk} x(t) + g_j(t, x(t)) \right) \right) + \left( \sum_{i=1}^r \sum_{j=1}^r h_i h_j B_i \left( D_{jk} x(t) + g_j(t, x(t)) \right) \right)^T P x(t) + \left( \sum_{i=1}^r \sum_{j=1}^r h_i h_j B_i \left( D_{jk} x(t) + g_j(t, x(t)) \right) \right) P \left( \sum_{i=1}^r \sum_{j=1}^r h_i h_j B_i \left( D_{jk} x(t) + g_j(t, x(t)) \right) \right) - (1 + v_k) x^T(t) P x(t) \leq 0 \text{ (13)}$$

By using Lemma 1 and substitute $L_{jk}(x) = D_{jk} x + g_j(t, x)$ in (13), (13) can be rewritten as

$$V(t^*_k, x(t)) - V(t_k, x(t)) - v_k d_k(V(t_k, x(t)))$$

$$= x^T(t) \left( \sum_{i=1}^r \sum_{j=1}^r h_i h_j B_i \left( D_{jk} x(t) + g_j(t, x(t)) \right) \right) + \left( \sum_{i=1}^r \sum_{j=1}^r h_i h_j B_i \left( D_{jk} x(t) + g_j(t, x(t)) \right) \right)^T P x(t) + \left( \sum_{i=1}^r \sum_{j=1}^r h_i h_j B_i \left( D_{jk} x(t) + g_j(t, x(t)) \right) \right) P \left( \sum_{i=1}^r \sum_{j=1}^r h_i h_j B_i \left( D_{jk} x(t) + g_j(t, x(t)) \right) \right) - (1 + v_k) x^T(t) P x(t) \leq 0 \text{ (13)}$$

By using Assumption 1 and Lemma 1, (14) is equal to the following

$$V(t^*_k, x(t)) - V(t_k, x(t)) - v_k d_k(V(t_k, x(t)))$$

$$= x^T(t) \left( \sum_{i=1}^r \sum_{j=1}^r h_i h_j B_i \left( D_{jk} x(t) + g_j(t, x(t)) \right) \right) + \left( \sum_{i=1}^r \sum_{j=1}^r h_i h_j B_i \left( D_{jk} x(t) + g_j(t, x(t)) \right) \right)^T P x(t) + \left( \sum_{i=1}^r \sum_{j=1}^r h_i h_j B_i \left( D_{jk} x(t) + g_j(t, x(t)) \right) \right) P \left( \sum_{i=1}^r \sum_{j=1}^r h_i h_j B_i \left( D_{jk} x(t) + g_j(t, x(t)) \right) \right) - (1 + v_k) x^T(t) P x(t) \leq 0 \text{ (13)}$$

By using Schur’s complement and considering (15), we have

$$\begin{bmatrix} O_{jk} & * & * & * \\ P & -P & \ast & \ast \\ B_i D_{jk} & 0 & \ast & \ast \\ I & 0 & 0 & \left( \left( \frac{2}{\epsilon} + r^2 \lambda_{\min} \right)^{-1} \right) I \end{bmatrix} < 0 \text{ (16)}$$

while yields $V(t^*_k, x(t)) - V(t_k, x(t)) - v_k d_k(V(t_k, x(t))) \leq 0$. Here, $O_{jk} = P B_i D_{jk} + D_i B_i^T P - (1 + v_k) P$, $T = P_1$. By pre- and post-multiplying (16) by $\text{diag}(P_1^{-1}, P, I, I)$ and with substitute $V_{ijk} = B_i D_{jk} T$, $O_{ijk} = B_i D_{jk} T + P D_{jk} B_i^T T - T (1 + v_k) \lambda_{\min} = \lambda_{\min} (B_i^T P B_i)$, it can be observed that following condition implies (10c).

$$\begin{bmatrix} O_{ijk} & * & * & * \\ T & -P & \ast & \ast \\ V_{ijk} & 0 & \ast & \ast \\ T & 0 & 0 & \left( \left( \frac{2}{\epsilon} + r^2 \lambda_{\min} \right)^{-1} \right) I \end{bmatrix} < 0 \text{ (17)}$$
Thus, the origin of the PDC-based T-S fuzzy impulsive control system (6) is stable and the proof is completed.

**Remark 4:** In some cases to reduce calculations, equation (11) can be rewritten as follows:

\[
D^+V(t,x) = \frac{\xi_k}{\Delta t_k} x^T P x
\]

\[
+ \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j x^T T A_i T + A_i T + 2 T F_i^T B_i^T - \frac{\xi_k}{\Delta t_k} T x
\]

\[
= \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j x^T (\tilde{T}_{ijk}) x \leq \sum_{i=1}^{r} h_i^2 x^T \tilde{T}_{ijk} x
\]

\[
+ 2 \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j x^T \left( \frac{\tilde{T}_{ijk}}{2} + \frac{P_{ijk}}{2} \right) x < 0
\]

(18)

Similar to (11), if \( \tilde{T}_{ijk} < 0 \) then \( D^+V(t,x) - \frac{\xi_k}{\Delta t_k} x^T P x < 0 \).

In the following corollary, the PDC-based fuzzy impulsive control system (6) with linear controller \( (g_j(t,x) = 0) \) is considered. In the case of \( I_{jk}(x) = D_{jk} x \), our task is to design \( \{u_k, D_{jk}\} \) and \( F_i^T \)'s such that PDC-based fuzzy impulsive control system (6) is stabilized.

**Corollary 1:** The origin of the PDC-based T-S fuzzy impulsive control system (6) is stable if there exist a positive-definite matrix \( P \), scalars \( \xi_k \)'s, \( v_k \)'s, positive scalars \( \Delta t_k \)'s, and matrices \( D_{jk} \)'s such that (10a), (10b) and the following conditions hold:

\[
\begin{pmatrix}
\tilde{H}_{ijk} & \ast \\
V_{ijk} - T
\end{pmatrix} \leq 0
\]

(19)

All notations used in here are similar to Theorem 2.

**Proof:** According to the aforementioned descriptions and similar to the proof of Theorem 2, it can be shown that:

\[
(t_k^+, x(t) + I_{k}(x(t))) - V(t_k, x(t))
\]

\[
- v_k d_k \left( V(t_k, x(t)) \right)
\]

\[
= x^T(t) P I_{k}(x(t)) + I_{k}^T(x(t)) P x(t)
\]

\[
+ I_{k}^T(x(t)) P I_{k}(x(t)) - x^T(t) P x(t)
\]

\[
= x^T(t) P \left( \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j B_i I_{jk}(x(t)) \right)
\]

\[
+ \left( \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j B_i I_{jk}(x(t)) \right)^T P x(t)
\]

\[
+ \left( \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j B_i I_{jk}(x(t)) \right)^T P \left( \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j B_i I_{jk}(x(t)) \right)
\]

\[
- (1 + v_k) x^T(t) P x(t) \leq 0
\]

Here, \( I_{jk}(x(t)) = D_{jk} x(t) \). So, we obtain

\[
V(t_k^+, x(t) + I_{k}(x(t))) - V(t_k, x(t))
\]

\[
- v_k d_k \left( V(t_k, x(t)) \right)
\]

\[
= x^T(t) P \left( \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j B_i I_{jk}(x(t)) \right)
\]

\[
+ \left( \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j B_i I_{jk}(x(t)) \right)^T P x(t)
\]

\[
+ \left( \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j B_i I_{jk}(x(t)) \right)^T P \left( \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j B_i I_{jk}(x(t)) \right)
\]

\[
- (1 + v_k) x^T(t) P x(t) \leq 0
\]

(20)

By using Schur’s complement, the following inequality is sufficient condition to get (21)

\[
\begin{pmatrix}
\tilde{H}_{ijk} & \ast \\
V_{ijk} - T
\end{pmatrix} \leq 0
\]

(22)

where \( \tilde{H}_{ijk} = P B_i D_{jk} + D_{jk}^T B_i^T P - (1 + v_k) P \), \( T = P^{-1} \).

By pre- and post-multiplying (22) by \( \text{diag}(P^{-1}, I) \) and by substituting \( V_{ijk} = B_i D_{jk} T \), \( \tilde{H}_{ijk} = B_i D_{jk} T + T D_{jk}^T B_i^T T - T(1 + v_k) \), we get

\[
\begin{pmatrix}
\tilde{H}_{ijk} & \ast \\
V_{ijk} - T
\end{pmatrix} \leq 0
\]

(23)

thus, the proof is completed.

Now, consider the following PDC-based fuzzy impulsive control system without considering PDC-based state feedback controller. \( (u_1(t) = 0) \)

\[
\begin{pmatrix}
\hat{x} = \sum_{i=1}^{r} h_i A_i \hat{x} \\
\Delta x|_{t_k} = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j B_i I_{jk}(x(t)) , t \neq t_k \\
x(t(t_k^+)) = x_0 , k = 1, 2, 3, ...
\end{pmatrix}
\]

The stability conditions of PDC-based fuzzy impulsive control system (24) with non-linear impulsive function \( I_{jk}(x) = D_{jk} x + g_j(x(t), x) \), can be summarized by the following corollary.

**Corollary 2:** The origin of the PDC-based T-S fuzzy impulsive control system (24) is stable if there exist a positive-definite matrix \( P \), scalars \( \xi_k \)'s, \( v_k \)'s, positive scalars \( \Delta t_k \)'s, and matrices \( D_{jk} \)'s such that (10a), (10c) and the following conditions hold:

\[
\psi_{ik} < 0
\]

(25)

where, \( \psi_{ik} = TA_i^T + A_i T - \frac{\xi_k}{\Delta t_k} T . \)

All notations used in here are similar to Theorem 2.
Proof: Consider the Lyapunov function \( V(t,x) = x^T(t)Px(t) \) for the PDC-based T-S fuzzy impulsive control system (24) with nonlinear controller \( I_{jk}(x) \). Now, from condition (1) of Theorem 1 and by taking the upper right-hand generalized derivative of the Lyapunov along the trajectories of system (24), the following is obtained

\[
D^+V(t,x) = \sum_{i=1}^{r} h_i(x(t))^T P x(t) + \sum_{i=1}^{r} h_i x^T(t) P A_i x(t) - \frac{\xi_k}{\Delta t_k} x^T(t) P x(t)
\]

Thus, the proof is completed.

In the previous corollary, a non-linear impulsive function was used. Now, we consider a linear impulsive function as \( I_{jk}(x) = D_{jk} x \). Corollary 3 is given to analysis the stability of PDC-based T-S fuzzy impulsive control system (24) with linear impulsive function \( I_{jk}(x) = D_{jk} x \).

Corollary 3: The origin of the PDC-based T-S fuzzy impulsive control system (24) is stable if there exist a positive definite matrix \( P \), scalars \( \xi_k \), \( v_k \)'s, positive scalars \( \Delta t_k \)'s, and matrices \( D_{jk} \)'s such that (10a), (19) and (25) hold.

All notations are similar to theorem2.

Proof: The proof is similar to the proof of Corollary 2. The only variation is that, here \( g_j = 0 \). Thus, the proof is completed.

Remark 5: The advantage of the suggested method in this work is that it gives more insight, flexibility, generality and simplicity regarding the stabilization of T-S fuzzy impulsive systems than the other methods [26], which apply a linear form of impulsive function.

4. Numerical Simulations

In this section, we present a design example to show how to perform the impulsive fuzzy control on the Lotka-Volterra predator-prey system. Consider Lotka-Volterra predator-prey system while is expressed with the following differential equation.

\[
\begin{align*}
\dot{x}_1(t) &= x_1(t)(\mu_1 - r_2 x_2(t)) \\
\dot{x}_2(t) &= x_2(t)(-\mu_2 - r_2 x_1(t))
\end{align*}
\]  

where \( x_1(t) \), \( x_2(t) \) (\( x_1(t) > 0 \), \( x_2(t) > 0 \)) denote the species density of the prey and the predators in the group at time \( t \), respectively. The coefficient \( \mu_1 \) and \( \mu_2 \) denote the birth rate of the preys and the death rate of the predators, respectively. The other two coefficients \( r_1 \) and \( r_2 \) (both positive) describe interactions between the species. System (27) can be rewritten as follows

\[
\begin{align*}
\dot{x}(t) &= A x(t) + \Phi(x(t))
\end{align*}
\]  

where \( A = \begin{bmatrix} \mu_1 & 0 \\ 0 & -\mu_2 \end{bmatrix} \) and \( \Phi(x) = \begin{bmatrix} -r_1 x_1 x_2 \\ -r_2 x_1 x_2 \end{bmatrix} \). By using the methods introduced in [27], we can obtain the following fuzzy impulsive control for the above predator-prey model as

\[
(\dot{x}(t) = A_i x(t) + B_i u_i(t), \quad t \neq t_k)
\]

Thus, the proof is completed.
The simulation result with the following conditions for Corollary 3 is shown in Figs. 3-4. The following parameters are applied which have been obtained by the Matlab LMI toolbox.

If $\Delta t_k = 0.1$

\[
P = \begin{bmatrix} 9.3876 & -0.0286 \ 0.0286 & 10.0822 \end{bmatrix}
\]

\[
\varepsilon = 0.1
\]

If $\Delta t_k = 0.01$

\[
P = \begin{bmatrix} 98.0533 & -0.0288 \ 0.0288 & 98.7623 \end{bmatrix}
\]

\[
D_{2k} = D_{1k} = [-0.2497 \ -0.2515 \]

\[
\varepsilon = 0.1
\]

For the sake of comparison, the responses for the two case ($\Delta t_k = 0.1$ and $\Delta t_k = 0.01$) have been shown.

**Remark 6:** According to Figs 1-4 as we expected, it is clear that the nonlinear fuzzy impulsive controller with PDC-based state feedback controller (Theorem 2) is far better than the linear fuzzy impulsive controller (Corollary 3). In addition, the proposed method for fuzzy impulsive controller coefficients calculation in this paper is more efficient than the methods proposed in other papers [26], which this is easily perceived with making comparisons between this paper and the others.

**Conclusion**

The impulsive control technique, which was proved to be suitable for complex and nonlinear system with impulsive effects, was applied to analyzing the framework of the fuzzy systems based on T-S model. First, the overall impulsive fuzzy system was obtained by blending local linear impulsive system. Meanwhile, the stabilization of the impulsive fuzzy system was derived by extended Lyapunov theory. Secondly, a numerical example for predator-prey systems with impulsive effects was given to illustrate the application of fuzzy impulsive control.
References


