Optimal Censor-based Detection in Cooperative Spectrum Sensing

M. Modarres-Hashemi¹, E. Berenjkoub², and N. Janatian³

¹,²,³ Department of Electrical and Computer Engineering, Isfahan University of Technology, Isfahan, Iran

Emails: modarres@cc.iut.ac.ir, e.berenjkoub@ec.iut.ac.ir, n.janatian@ec.iut.ac.ir

Abstract: Cooperative spectrum sensing is one of the most important proposed methods to combat with fading, shadowing, and hidden primary user problems. Also, censoring idea was proposed to reduce the communication overhead in cooperative spectrum sensing. In this paper, optimal censor-based strategies are investigated. Optimization problem is formulated under the bandwidth constraint for fixed local rule scenario, fixed fusion rule (OR) scenario, and also joint optimization of local and fusion rules. Solutions to the first two problems are found analytically and the joint optimization problem is solved by means of an iterative search algorithm. Simulation results are also presented to study the impact of the communication constraint on the censoring scheme and the detection performance.

Keywords: Cooperative spectrum sensing, Censoring, Detection

1. Introduction

The rapid growth in wireless communications over the last decades has contributed to a huge demand on the deployment of new wireless services in both the licensed and unlicensed frequency spectrum. However, recent studies show that the fixed spectrum assignment policy enforced today results in poor spectrum utilization. Upon this scenario, cognitive radio (CR) [1] has emerged as a promising technology to enable the access of unoccupied frequency bands, called white space or spectrum holes, and thereby increase the spectral efficiency. The fundamental task of each CR user is to detect the licensed users, also known as primary users (PU). This is accomplished in a process called spectrum sensing.

The spectrum sensing problem has been studied extensively in the literature. In general, spectrum sensing methods can be categorized into energy detection, matched filter detection and cyclostationary feature detection [2]. In practice, many factors such as multipath fading, shadowing, and the hidden PU problem [3] may significantly affect the detection performance in spectrum sensing. To overcome these limitations, cooperation in the spectrum sensing has been proposed. The main idea behind the cooperative spectrum sensing (CSS) is to enhance the detection performance by exploiting the spatial diversity in the observations of spatially located CR users. Through this cooperation, the CR users can share their local sensing data with each other or report it to a fusion center (FC) [3]. Due to the constraint of reporting channel bandwidth, the local sensing data should be quantized before transmitting to FC. In the case of a large number of users, the bandwidth required for reporting becomes huge. In order to reduce the sharing bandwidth, local observations of cognitive radios are quantized to one bit (hard decisions) [4].

Another approach to reduce the reporting channel's bandwidth is censoring. Censoring idea was proposed by Rago et. al [5-6] in 1993 in the context of distributed detection in sensor networks. In this scheme, sensors either send or do not send some real-valued function of their observation to a FC based on a communication rate constraint. The work in [6] shows that in the censoring scenario with independent sensors’ observations, it is optimal (in both the Bayesian and the Neyman-Pearson sense) for the sensors to not send their likelihood ratios (LRs) if they fall in a particular interval, and that the optimal censoring region is a single interval \((t_1, t_2)\) of the likelihood ratio.

To obtain the optimum censoring thresholds and the optimum decision rule in the censoring formulation, joint optimization is required across the sensors and the FC. Further simplification is provided in [7-8] by finding the general conditions under which the optimal lower threshold \(t_1\) of censoring is zero. In this case, called on/off signalling [9], whenever the local LR exceeds \(t_2\), the sensor transmits the LR to FC; otherwise the sensor remains silent. [9-11] have considered on/off signalling scheme to its extreme case; if the local statistic (LR, Energy detector and etc.) exceeds \(t_2\), the sensor sends only a single bit, indicating that the statistic falls into the “send” region. Fusion of censored decisions transmitted over fading reporting channels is studied in [9-11] assuming local energy detector with known upper threshold \(t_2\). Fading channels are also modelled by Rayleigh [9], Ricean [10] and Nakagami [11] distributions. Also, [12-13] design the local sensor decision rules (in terms of the LR threshold) in the same scenario.
The idea of censoring was first applied to cooperative spectrum sensing in the cognitive radio network in 2007 [14]. Most of the work done in this area concentrates on the analysing the detection performance of censoring strategy in a fixed local and fusion rule. In [14], the censored energy detection using the hard decision “OR” rule in FC for both perfect and imperfect control channels was analysed. In [15], Rayleigh fading in sensing channel is also considered and the optimal thresholds for censoring region are found by minimizing the communication overhead under a constraint of spectrum utilization. The performances of the censored energy detectors with two pre-determined thresholds have been also analyzed by considering both soft decision and hard decision rules in [16], [17] and [18] proposed censoring for collaborative spectrum sensing based on cyclostationarity detection, and autocorrelation detection, respectively.

In this paper, optimal censor-based strategies are investigated in Neyman-Pearson (NP) sense. Optimization problem is formulated under the bandwidth constraint for fixed local rule scenario, fixed fusion rule (OR) scenario in FC and joint optimization of local and fusion rules. The last problem is a considerably harder problem compared to the first two problems as it involves design of distributed sensors, as opposed to the fusion rule study where a centralized processing is carried out. Solutions to the last two problems are found analytically and the joint optimization problem is solved by means of an iterative search algorithm.

The organization of the paper is as follows. In the next section, we introduce the system model and assumptions. In section 3, an optimal fusion rule is derived in NP sense, assuming pre-fixed (\(t_1, t_2\)) in local CRs. Then local rules are optimised in an adhoc case in which OR rule is used at FC. Finally the local and fusion rules are optimized jointly. Simulation results are provided in section 4 and finally the paper ends with conclusion.

2. System Model

In this section, we describe the system model of censor-based cooperative spectrum sensing. Suppose that we have \(N\) “independent” CRs whose observations are statistically independent when conditioned on whether hypothesis \(H_0\) (absence of PU) or hypothesis \(H_1\) (presence of PU) is true. Signal transmitted from PU to CR’s is assumed to be unknown deterministic. For the spectrum band of interest, the received signal sample of each CR under each hypothesis can be represented as:

\[
H_0: y(k) = n(k) \\
H_1: y(k) = h s(k) + n(k)
\] (1)

where \(s(k)\) denotes the PU signal, \(n(k)\) is zero mean white Gaussian noise \((-\mathcal{N}(0, \sigma^2))\), \(h(k)\) is the sensing channel (PU to CR channel) impulse response which is assumed to be unchanged during the sensing process and \(k = 0, 1, \ldots, m − 1\) is the sample index.

Every CR conducts local spectrum sensing independently based on its observations. For simplicity, we use energy detection in spectrum sensing and only consider the case where users send their 1-bit decisions regarding the spectrum occupancy rather than their decision statistic which is denoted by \(Y\):

\[
Y = \frac{1}{N_0} \int_0^T y^2(t) dt \approx \frac{1}{2W N_0} \sum_{i=1}^m y(i)^2
\] (2)

where \(N_0\) denotes the one-sided noise PSD (Power Spectral Density) and \(W\) is the signal bandwidth.

A single interval \((t_1, t_2)\) is selected as a 'No send' region and if the local statistic \(Y\) falls into \((t_1, t_2)\) then it will send nothing; otherwise a binary local decision is sent to FC as shown in Fig.1. Furthermore, we assume that all CRs use the same thresholds.

<table>
<thead>
<tr>
<th>Decision (H_0)</th>
<th>No Send</th>
<th>Decision (H_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(t_1)</td>
<td></td>
<td>(t_2)</td>
</tr>
</tbody>
</table>

Fig. 1: Censoring strategy

Based on the above, optimal local rules can be determined completely by means of optimal thresholds \(t_1\), \(t_2\). For a nonfading environment, false alarm and detection probability in each node can be formulated as [4]:

\[
P_f = \Pr(Y > t_2 | H_0) = \int_{t_2}^{\infty} f_{Y|H_0}(y)dy
\]

\[
= \frac{\Gamma(m/2, t_2/2)}{\Gamma(m/2)}
\] (3)

\[
P_d = \Pr(Y > t_2 | H_1) = \int_{t_2}^{\infty} f_{Y|H_1}(y)dy
\]

\[
= Q\left(\frac{\sqrt{2\gamma}}{\sqrt{\lambda}}, \sqrt{\gamma}\right)
\] (4)

where \(\Gamma(m/2, t_2/2)\) is the incomplete gamma function, \(Q_{\alpha}(\sqrt{2\gamma}, \sqrt{\lambda})\) is the generalised Marcum Q-function and \(\gamma\) denotes the SNR.

In fading environment, average probability of detection can be derived by averaging (4) over fading statistics:

\[
P_d = \int_{-\infty}^{\infty} P_d(y) f(y) dy
\] (5)

Under Rayleigh fading, \(y\) would have an exponential distribution \(f(y) = \frac{1}{\gamma} \exp\left(-\frac{y}{\gamma}\right)\). In this case, a closed-form formula for \(P_d\) may be obtained by substituting \(f(y)\) in (5) [4]:

\[
P_d = e^{-\lambda} \sum_{n=0}^{m/2-2} \frac{1}{n!} \left(\frac{t_2}{\gamma}\right)^n
\]
\[
+ \left( \frac{1 + \gamma}{\gamma} \right)^{m/2-1} \left[ e^{-\frac{t_2}{2(1+\gamma)}} e^{-\frac{1}{2} \sum_{n=0}^{m/2-2} \frac{1}{n!} \left( \frac{t_2 \gamma}{2(1+\gamma)} \right)^n} \right]
\]

(6)

Final decision will be made by FC based on local decisions reported. In this study, reporting channels (CR to FC channel) are assumed to be independent and error free.

3. Optimal Censor-Based Strategies

In this section, first an optimal fusion rule is derived in NP sense, assuming pre-fixed \((t_1, t_2)\) in local CR’s. Then local rules are optimised under bit rate constraint in an adhoc case in which OR rule is used at FC in section 3.2. Finally the local and fusion rules are optimized jointly in section 3.3.

3.1 Optimal fusion rule

In this case, the NP optimization problem can be written as:

\[
\max_{Q_d} Q_f \quad \text{s.t.} \quad Q_f \leq \overline{Q}_f
\]

(7)

where \(Q_d\) and \(Q_f\) are the global probabilities of detection and false alarm, respectively. Solution to this NP problem is proved to be a threshold test based on likelihood ratio as follows [19]:

\[
L_{FC}(U) = \frac{Pr(U = 1 | H_1)}{Pr(U = 1 | H_0)} \frac{H_1}{H_0} \quad \text{where} \quad \frac{H_1}{H_0} \geq \eta
\]

(8)

In which, \(U = \{U_1, U_2, ..., U_N\}\) is the vector of received bits in FC, \(U_i \in \{0, 1, \emptyset\}\) (\(\emptyset\) declares no send case) and \(\eta\) is selected to achieve the desired constant false alarm probability, \(\overline{Q}_f\). Assume that FC receives \(K\) bits out of \(N\), with \(M\) ‘1’ bits out of \(K\), so we have:

\[
Pr(U = 1 | H_1) = \beta_1 M \beta_0^{K-M} \alpha^K_0 \]

and

\[
Pr(U = 1 | H_0) = \alpha M \alpha^K_0 \alpha^K_0 \]

(9)

(10)

where \(\beta_j = Pr(\text{‘}j\text{’ is transmitted}|H_1)\) and \(\alpha_j = Pr(\text{‘}j\text{’ is transmitted}|H_0)\). Substituting (9) and (10) in (8) yields:

\[
L_{FC}(U) = \frac{Pr(U = 1 | H_1)}{Pr(U = 1 | H_0)} \frac{H_1}{H_0} \geq \eta
\]

(11)

By defining the following probabilities:

\[
\alpha_1 = Pr(Y > t_2 | H_0) = P_{f_1} \\
\alpha_0 = Pr(Y < t_1 | H_0) = 1 - P_{f_1} \\
\alpha_\phi = Pr(t_1 < Y < t_2 | H_0)
\]

\[
= Pr(Y < t_2 | H_0) - Pr(Y < t_1 | H_0) \\
= P_{f_1} - P_{f_2} \\
\beta_1 = Pr(Y > t_2 | H_1) = P_{d_2} \\
\beta_0 = Pr(Y < t_1 | H_1) = 1 - P_{d_1} \\
\beta_\phi = Pr(t_1 < Y < t_2 | H_1) = P_{d_1} - P_{d_2}
\]

(14)

(15)

(16)

(17)

(11) can be written as:

\[
L_{FC}(U) = \frac{(P_{d_2}^{M}) (1 - P_{d_1}) (P_{d_1} - P_{d_2})^{N-K} H_1}{P_{f_2} (1 - P_{f_1})} \geq \eta
\]

(18)

This can be simplified as follows [20]:

\[
M = \sum_{i \in \{1, 2, ..., K\}} u_i
\]

\[
H_1 \ln \left( \frac{P_{f_1} - P_{d_1}}{P_{f_2} - P_{d_2}} \right)^{N-K} \left( 1 - P_{d_1} \right)^K \leq \eta \]

\[
H_0 \ln \left( \frac{P_{f_1} - P_{d_1}}{P_{f_2} - P_{d_2}} \right)^{N-K} \left( 1 - P_{d_1} \right)^K \leq \eta \]

(19)

which is equivalent to counting rule with a threshold depending on the number of received bits, \(K\). Therefore, \(Q_d\) and \(Q_f\) can be given by (see Appendix.1) [20]:

\[
Q_d = \sum_{k=1}^{N} \left( \frac{N}{K} \right) (P_{d_1})^k \left( 1 - P_{d_1} \right)^{N-k} - P_{d_2} \sum_{k=1}^{K} \left( \frac{K}{k} \right) P_{d_1}^k (1 - P_{d_1})^{K-k} + S \left( P_{f_1} - P_{f_2} \right)^N
\]

(20)

\[
Q_f = \sum_{k=1}^{N} \left( \frac{N}{K} \right) (P_{f_1} - P_{f_2})^k \left( 1 - P_{f_1} \right)^{N-k} + S \left( P_{f_1} - P_{f_2} \right)^N
\]

(21)

where \(S\) indicates whether we decide on \(H_0\) or \(H_1\) if no bit is sent to FC (i.e. \(K = 0\)):

\[
S = \begin{cases} 
0 & \frac{(P_{d_1} - P_{d_2})^N}{(P_{f_1} - P_{f_2})^N} < \eta \\
1 & \frac{(P_{d_1} - P_{d_2})^N}{(P_{f_1} - P_{f_2})^N} > \eta 
\end{cases}
\]

(22)

3.2 Optimal local detection with OR fusion rule

In this section, local thresholds \(t_1, t_2\) are optimized assuming OR fusion rule in FC. This assumption in FC is equivalent to \(\eta_K = 1\) in equations (20) and (21) and so we have:

\[
\frac{(P_{d_1} - P_{d_2})^N}{(P_{f_1} - P_{f_2})^N} < \eta
\]

(23)
\[ Q_{f,OR} = \sum_{K=1}^{N-K} \binom{N}{K} (P_{f_1} - P_{f_2})^{N-K} \sum_{i=1}^{K} \binom{K}{i} P_{f_2}^i (1 - P_{f_1})^{K-i} \label{eq:QfOR} \]

\[ + S(P_{f_1} - P_{f_2})^N \]  

\[ Q_{d,OR} = \sum_{K=1}^{N} \binom{N}{K} (P_{d_1} - P_{d_2})^N \]  

\[ - P_{d_2}^{N-K} \sum_{i=1}^{K} \binom{K}{i} P_{d_2}^i (1 - P_{d_1})^{K-i} \]  

\[ + S(P_{d_1} - P_{d_2})^N \]  

\[ \text{Assuming } S = 1, \text{ these equations can be simplified to } [20]: \]

\[ Q_{f,OR} = 1 - (1 - P_{f_1})^N + (P_{f_1} - P_{f_2})^N \]  

\[ Q_{d,OR} = 1 - (1 - P_{d_1})^N + (P_{d_1} - P_{d_2})^N \]  

Therefore, the optimization problem can be written as:

\[ \max_{t_1,t_2} Q_{d,OR} \]  

\[ \text{s.t. } Q_f \leq \overline{Q}_f \]  

\[ \sum_{i=1}^{N} P(Y_i \notin (t_1, t_2)|H_0) \leq \kappa_{NP} \leq N \]  

where \( \sum_{i=1}^{N} P(Y_i \notin (t_1, t_2)|H_0) \) is the average communication constraint in NP criterion. In our scenario, second constraint in (27) can be written as:

\[ \alpha_1 + \alpha_0 = 1 - P_{f_1} + P_{f_2} \leq \frac{\kappa_{NP}}{N} \leq \kappa'_{NP} \]  

**Lemma 1.** For any fixed value of \( P_{f_1}, Q_{d,OR} \) is an increasing function of \( P_{f_2} \) [20] (see Appendix 2).

Therefore, given \( P_{f_1} \), the optimum is achieved when the following equation stands:

\[ 1 - \kappa'_{NP} + P_{f_2} = P_{f_1} \]  

**Lemma 2.** Substituting (29) in (25) and (26), \( Q_{d,OR} \) and \( Q_{f,OR} \) will be increasing functions of \( P_{f_2} \) [20] (see Appendix 3).

According to Lemma 2, optimal solution is achieved when:

\[ Q_{f,OR} = 1 - (1 - P_{f_2})^N + (P_{f_1} - P_{f_2})^N = \overline{Q}_f \]  

Solving (29) and (30) at the same time, results in \( P_{f_1}, P_{f_2} \). Therefore optimal \( t_1, t_2 \) can be found using (12) and (13).

### 3.3 Joint Optimization of local and fusion rule

In this section, the local and fusion rules are optimized jointly:

\[ \max Q_d \]  

\[ \text{s.t. } Q_f \leq \overline{Q}_f \]  

\[ \sum_{i=1}^{N} P(Y_i \notin (t_1, t_2)|H_0) \leq \kappa_{NP} \leq N \]  

In which the second constraint can be simplified to (28). Combining (28) with \( 0 \leq P_{f_1}, P_{f_2} \leq 1 \) results in:

\[ 0 \leq P_{f_2} \leq \kappa'_{NP} \]  

\[ 1 - \kappa'_{NP} + P_{f_2} \leq P_{f_1} \leq 1 \]  

At any rate, under this censoring scheme, the optimal fusion rule is (and must be) a likelihood ratio test over the received and non received LR’s given by (11) [6], thereby optimization problem is simplified to [20]:

\[ \max Q_d \]  

\[ \text{s.t. } Q_f \leq \overline{Q}_f \]  

\[ 0 \leq P_{f_2} \leq \kappa'_{NP} \]  

\[ 1 - \kappa'_{NP} + P_{f_2} \leq P_{f_1} \leq 1 \]  

In which \( Q_d \) and \( Q_f \) are given in (20) and (21). Finding analytical solution to this problem is not easy due to complex relation of \( Q_d \) and \( Q_f \) with \( P_{f_1} \) and \( P_{f_2} \). Consequently, an iterative search method is used to find the optimal thresholds[20].

**Iterative method**

**Step 1** Choose an error tolerance level, \( \delta \)

**Step 2** Let \( P_{f_2} = \delta \)

**Step 3** Compute the corresponding \( t_2 \) and \( P_{d_2} \) from (12) and (15) (using (3) and (4))

**Step 4** Let \( P_{f_1} = P_{f_2} \)

**Step 5** Compute the related \( t_1 \) and \( P_{d_1} \) from (13) and (16)

**Step 6** For assumed \( P_{f_1}, P_{f_2} \) and calculated \( P_{d_1}, P_{d_2} \), calculate \( \eta \) that satisfies \( Q_d = \overline{Q}_f \) given in (21)

**Step 7** Compute \( Q_d \) from (20)

**Step 8** Save \( Q_d \) and the corresponding \( t_1, t_2, \eta \)

**Step 9** Check if \( P_{f_1} \leq 1 \)

- Yes: Let \( P_{f_1} = P_{f_2} + \delta \) and go to step 5
- No: Go to step 10

**Step 10** Check if \( P_{f_2} \leq \kappa'_{NP} \)

- Yes: Let \( P_{f_2} = P_{f_2} + \delta \) and go to step 3
- No: find the maximum value of \( Q_d \), the corresponding \( t_1, t_2, \eta \) are the optimal thresholds; Stop.

### 4. Simulation Results

In this section, we will demonstrate the performance of the proposed censor-based rule by means of simulation. Fig.2 provides plots of ROC curve (\( Q_d \) versus \( Q_f \)) under AWGN and Rayleigh fading scenario for different number of CRs. Average SNR(\( \overline{\gamma} \)) and number of samples(\( m \)) are assumed to be 5 dB and 10, respectively. Error tolerance (\( \delta \)) and communication constraint (\( \kappa'_{NP} \)) are also set to .01 and .9, respectively.
As expected, increasing the number of CR nodes results in significant improvement in detection performance. In particular, for a probability of false alarm equal to 0.01, probability of detection in cooperative sensing with \( N = 2 \) is only .2 while it changes to more than .95 in \( N=20 \).

Fig. 3 Compares the performance of optimal fusion rule investigated in section 3.3 with 'OR' rule discussed in section 3.2. Simulation parameters are chosen as before. As seen in this figure, performance of OR rule is close to optimal at lower number of CR nodes. However, its performance degrades relative to optimal fusion rule at higher number of nodes such as \( N=10, 20 \).

Fig. 4 compares the censor-based optimal method for \( \kappa'_{NP} = .2, .5, 1 \) with optimal binary case (no censor). In this simulation, \( \bar{y} \) and \( N \) are assumed to be 3 dB and 5, respectively. Other parameters are set as before. The optimal fusion rule in binary case is counting rule [21] and in this scenario, local and fusion rule thresholds are found numerically to achieve the maximum \( Q_d \) in a fixed \( Q_f \). It can be seen that the performance of censor-based method for \( \kappa'_{NP} = .5, 1 \) is slightly better than binary case which is due to more information we have in censor-based method since we use two thresholds in each node. It should be noted that censor-based method with \( \kappa'_{NP} = 1 \) is different from binary case. Since in the latter, data rate is equal to 1, however in the former, the average data rate is less than or equal to 1. For \( \kappa'_{NP} = .2 \), their performance is very close to each other. This means that even for severe constraint on bit rate, performance of censor-based method is acceptable.

5. Conclusion

In this paper, we studied censor-based cooperative spectrum sensing in CR network. Optimization problem was formulated and solved under the bandwidth constraint for fixed local rule scenario, fixed fusion rule (OR) scenario, and joint optimization of the local and fusion rules. Simulation results were also presented to study the impact of the communication constraint on the censoring scheme and the detection performance.

It was seen that using the optimal censor-based strategy, spectrum sensing performance will not be degraded with respect to binary (no censor) scenario even in severe communication constraints. As a result, censoring method can be safely used instead of binary method under bandwidth limitations; albeit at the expense of greater complexity in the system design.

Appendix 1.

Global probability of false alarm can be written as [19]:

\[
Q_f = \Pr(L_{FC} > \eta|H_0)
\]

\[
= \sum_{K=0}^{N} (\Pr(L_{FC} > \eta|H_{0}, K \text{ bits are transmitted}) \cdot \Pr(K \text{ bits are transmitted}|H_0))
\]

\[
= \sum_{K=0}^{N} (\Pr(L_{FC} > \eta|H_{0}, K \text{ bits are transmitted}) \cdot \Pr(K \text{ bits are transmitted}|H_0))
\]

where,

\[
\Pr(K \text{ bits are transmitted}|H_0) = \binom{N}{K} (\alpha + \alpha_2)^K (\alpha_2)^{N-K} = \binom{N}{K} (1 - P_{f_1} + P_{f_2})^K (P_{f_1} - P_{f_2})^{N-K}
\]

(36)

According to (19), the number of '1's out of \( K \) received bits should be at least \( \lceil K \eta \rceil \) '1's out of ' should be received at FC to have \( L_{FC} > \eta \). Therefore,

\[
\Pr(L_{FC} > \eta|H_0, K \text{ bits}) =
\]
\[
\sum_{i=\lceil n/2 \rceil}^{K} \binom{K}{i} P_i \Pr(i \text{ nodes send } '1' | H_0, K \text{ bits}) = \sum_{i=\lceil n/2 \rceil}^{K} \binom{K}{i} P_i (1 - P_i)^{K-i} \frac{1}{(1 - P_f + P_f)^K}, K > 0
\]

So we have,

\[
Q_f = \sum_{k=0}^{N} \Pr(L_{FC} > \eta | H_k, K \text{ bits}) \Pr(K | H_d) + \Pr(0 \text{ bits} | H_d) \Pr(L_{FC} > \eta | H_0, 0 \text{ bits}) = \sum_{k=1}^{N} \binom{N}{k} (P_{f_1} - P_{f_2})^{N-k} \sum_{i=\lceil n/2 \rceil}^{K} \binom{K}{i} P_i (1 - P_i)^{K-i} + S(P_{f_1} - P_{f_2})^N
\]

Appendix 2.

Calculating derivative of \(Q_{d,OR}\) in a fixed \(P_{f_2}\) with respect to \(P_{f_1}\), we have:

\[
\frac{\partial}{\partial P_{f_1}} Q_{d,OR} = N \frac{dP_{d_2}}{dP_{f_1}} (1 - P_{d_2})^{-1} - N \frac{dP_{d_2}}{dP_{f_1}} (P_{d_1} - P_{d_2})^{-1} = N \frac{dP_{d_2}}{P_{f_1}} (1 - P_{d_2})^{-1} - (P_{d_1} - P_{d_2})^{-1} \quad (39)
\]

\[
\frac{dP_{d_2}}{dP_{f_1}} > 0 \quad (\text{since } P_{d_2} \text{ is an increasing function of } P_{f_2}) \quad \text{and} \quad (1 - P_{d_2})^{-1} - (P_{d_1} - P_{d_2})^{-1} > 0 \quad (\text{since } P_{d_1} < 1).
\]

This leads to \(\frac{\partial}{\partial P_{f_1}} Q_{d,OR} > 0\) and lemma 1 is proved.

Appendix 3.

Substituting (29) in (25), we have:

\[
Q_{f,OR} = 1 - (1 - (P_{f_1} - 1 + \kappa'_{NP}))^N + (1 - \kappa'_{NP})^N
\]

Calculating derivative of \(Q_{f,OR}\) and \(Q_{d,OR}\) with respect to \(P_{f_1}\), we have:

\[
\frac{\partial}{\partial P_{f_1}} Q_f = N \left(1 - (P_{f_1} - 1 + \kappa'_{NP})\right)^{N-1} = N(1 - P_{f_1})^{N-1} > 0 \quad (41)
\]

And,

\[
\frac{\partial}{\partial P_{f_1}} Q_d = N \frac{dP_{d_2}}{P_{f_1}} \frac{dP_{d_2}}{dP_{f_1}} (1 - P_{d_2})^{-1} + N \frac{dP_{d_2}}{dP_{f_1}} \frac{dP_{d_2}}{dP_{f_1}} (P_{d_1} - P_{d_2})^{-1} \quad (42)
\]

Since \(\frac{dP_{d_2}}{dP_{f_1}} = 1\) (from (29)), \(\frac{\partial}{\partial P_{f_1}} Q_d\) can be simplified to:

\[
\frac{\partial}{\partial P_{f_1}} Q_d = N \frac{dP_{d_2}}{P_{f_1}} ((1 - P_{d_2})^{-N-1} - (P_{d_1} - P_{d_2})^{-N}) + N \frac{dP_{d_2}}{P_{f_1}} (P_{d_1} - P_{d_2})^{-N} > 0 \quad (43)
\]

So, lemma 2 is proved.

References