A New State Estimation Method for Unit Time-Delay Systems Based on Kalman Filter

Behrouz Safarinejad*, Mohiyeddin Mozaffari**
* Department of Electrical Engineering, Shiraz University of Technology, safarinejad@sutech.ac.ir.
** Department of Electrical Engineering, Shiraz University of Technology, m.mozaffari@sutech.ac.ir.

Abstract: A new State estimation method for unit time-delay systems will be represented in this paper. Kalman filter will be considered to estimate the states of a linear system with unit time-delay. First, an augmented state vector is defined which consists of the main state vector and the state vector with a unit time-delay. The states of the augmented system will be estimated using Kalman filter algorithm. From the augmented state vector, the main state vector can be obtained. Finally, an example is provided to test the performance of the proposed method.

Keywords: Kalman filter, state estimation, augmented system, time delay.

1. Introduction

State space models have wide applications in many areas, e.g., system identification [1–3], adaptive filtering [4–6], system modelling and control [7,8], energy and power systems [9,10], signal processing [11,12]. Most contributions for systems with time-delay focus on the studies of control schemes. For example, Yan and Shi discussed the robust discrete-time sliding mode control for the uncertain systems with time-varying state delay [13], Shi and Yu studied the output feedback stabilization of the networked control systems with random delays modelled by Markov chains [14].

The Kalman filter has played a central role in system theory and has found wide applications in many fields such as control, signal processing, and communication [15]. There are various algorithms for state estimation such as Kalman filter, extended Kalman filter, unscented Kalman filter, optimal smoothing and particle filters. The basic idea in all of the mentioned methods is the Kalman filter. In this paper, due to linear dynamic of the system, Kalman filter will be used for state estimation.

This paper considers estimation problem of time-delay linear control systems based on the augmented system idea. The augmented system is a new-type state estimation and deals with estimation problems using augmented state vector. The basic idea is to define a new state vector including the main state vector and state vector with a unit time-delay. Then, by implementing Kalman filter algorithm for estimating the states, the main state vector can be obtained from the augmented state vector.

Briefly, this paper is organized as follows. In Section 2 an input–output representation related to the state space model with a unit time-delay is derived. Section 3 presents transfer function and input–output representation of augmented system. In Section 4 Kalman filter algorithm for augmented system that will be presented. Section 5 provides an illustrative example to verify the performance of the proposed algorithm. Finally, concluding remarks are given in Section 6.

2. Input-Output Representation of System with Unit Time-Delay

Consider the following state space model with a unit time-delay [16],

$$x(k+1) = Ax(k) + Bu(k-1) + gu(k)$$

$$y(k) = hx(k)$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}$ is the system input, and $y(k) \in \mathbb{R}$ is the system output. $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $g \in \mathbb{R}^m$, and $h \in \mathbb{R}^m$ are the system parameter matrices and vectors.

Since Equation (1) contains the term $x(k-1)$, we say that this state equation has a unit time-delay. In the following the state space model with time-delay in Equation (1) and Equation (2) will be transformed into an input–output representation and its transfer function will be obtained.

**Lemma 1.** For the state space model in Equation (1) and Equation (2), the transfer function from the input $u(k)$ to the output $y(k)$ is given by

$$T(q) = h(q^2 - Aq - B)^{-1} gq$$

where $q$ represents a unit forward operator that can be shown as

$$q^k x(k) = x(k+1) \quad \& \quad q^{-1} x(k) = x(k-1)$$

**Proof:** the proof is presented in [16].

Now, for a second-order system with the following matrices’ coefficients:
The augmented system is a fourth-order system with the following matrices’ coefficients

\[ F = \begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \end{bmatrix} \in \mathbb{R}^{2 \times 4}, \quad G = \begin{bmatrix} g_1 \\ g_2 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^{4 \times 1} \]

\[ H = \begin{bmatrix} h_1 & h_2 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{1 \times 4} \]

The transfer function from the input \( u(k) \) to the output \( y(k) \) will be

\[
T(q) = -g_2 \left( \frac{h_1(a_{11}q^2 + b_{11}q) - h_2(-q^2 + a_{12}q^2 + b_{12}q)}{\text{den}} \right) \\
- g_1 \left( \frac{h_1(a_{11}q^2 + b_{11}q) - h_2(-q^2 + a_{12}q^2 + b_{12}q)}{\text{den}} \right) \tag{11}
\]

\[
\text{den} = (b_{21}b_{22} - h_1b_{12}) + (a_{11}b_{22} + a_{22}b_{12} - a_{12}h_{12} - a_{22}h_{11})q \\
+ (h_1 + b_{22} - a_{11}a_{12} + a_{12}a_{11})q^2 + (a_{11} + a_{22})q^3 - q^4 \tag{12}
\]

3. Augmented System

In this section using the system dynamic with a unit time-delay a new state vector called augmented state vector is defined. Therefore an augmented system will be derived from the system with unit time-delay.

The augmented state vector is defined as:

\[
X(k) = \begin{bmatrix} x(k) \\ x(k-1) \end{bmatrix} \Rightarrow X(k+1) = \begin{bmatrix} x(k+1) \\ x(k) \end{bmatrix} \tag{7}
\]

where \( x(k) \) is the state vector of the main system and \( x(k-1) \) is the state vector with a unit time-delay. Then the system equations are rewritten as

\[ X(k+1) = FX(k) + Gu(k) \tag{8} \]

\[ y(k) = HX(k) + v(k) \tag{9} \]

where

\[
F = \begin{bmatrix} A & B \\ I_n & 0 \end{bmatrix} \in \mathbb{R}^{2n \times 2n}, \quad G = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \in \mathbb{R}^{2n \times 1} \\
H = \begin{bmatrix} h \\ 0 \end{bmatrix} \in \mathbb{R}^{1 \times 2n}
\]

Using the properties of the shift operator \( q \), the transfer function of the augmented system can be derived from the input \( u(k) \) to the output \( y(k) \) as

\[
T'(q) = H(qI - F)^{-1}G \tag{10}
\]

Now, for a second-order system with the following matrices’ coefficients

\[
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}^{2 \times 2}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \in \mathbb{R}^{2 \times 2} \\
g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \in \mathbb{R}^{2 \times 1}, \quad h = [h_1, h_2] \in \mathbb{R}^{2 \times 1}
\]

The dynamic system is given by the following equations:

\[ X(k+1) = FX(k) + Gu(k) + w(k) \tag{13} \]

\[ y(k) = HX(k) + v(k) \tag{14} \]

\[ E[w_iw_j^T] = Q_i \delta_{i-j} \tag{15} \]

\[ E[v_i,v_j^T] = R_i \delta_{i-j} \tag{16} \]

\[ E[v_iw_i^T] = 0 \tag{17} \]

where \( u(k) \) is a known input, \( w(k) \) and \( v(k) \) are white noise sequences with zero mean and variances \( Q \) and \( R \) respectively. \( w(k) \) is called process noise and \( v(k) \) is called measurement noise. In practice, the process noise covariance \( Q \) and measurement noise covariance \( R \) matrices might be time varying, however here we assume that they are constant.
2. The Kalman filter is initialized as follows:

Expected value of the initial state $X_0$
$$\hat{X}_0^* = E(X_0)$$ (18)

Covariance of initial estimate of $X_i$
$$P_0^* = E[(X_0 - \hat{X}_0^*)(X_0 - \hat{X}_0^*)^T]$$ (19)

3. The Kalman filter is given by the following equations, which are computed for each step $k=1, 2, \ldots$

$$P_k^* = F P_{k-1}^* F^T + Q_{k-1}$$ (20)

$$K_k = P_k^* H^T (HP_k^* H^T + R)^{-1}$$ (21)

$$\hat{X}_k^* = F \hat{X}_{k-1}^* + G u_{k-1}$$ (22)

$$\hat{X}_k^* = \hat{X}_k^* + K_k (y_k - H \hat{X}_k^*)$$ (23)

$$P_k^* = (I - K_k H) P_k^* (I - K_k H)^T + K_k R K_k^T$$ (24)

In the above from Equation (18) to Equation (22) in each time step we compute: estimation error covariance of $\hat{X}_k^*$, Kalman filter gain, a priori estimate of $X_0$, a posterior estimate of $X_k$, and the estimation error covariance of $\hat{X}_0^*$, respectively. In the mentioned algorithm, Equation (18) and Equation (20) form time update steps, and Equation (19), Equation (21) and Equation (22) form measurement update steps that are shown in Fig. 1.

The next section will be allocated to estimate the states of a second-order system with a unit time-delay by presenting an example.

5. Simulation Results

Consider the following second-order system
$$x(k+1) = \begin{bmatrix} 0 & 1.00 \\ -1.20 & -0.60 \end{bmatrix} x(k) + \begin{bmatrix} 0.20 & 0.30 \\ -0.30 & 0.40 \end{bmatrix} x(k-1)$$
$$+ \begin{bmatrix} 1.00 \\ 0.20 \end{bmatrix} u(k) + w(k)$$

$$y(k) = \begin{bmatrix} 1.20 & 1.75 \end{bmatrix} x(k) + v(k)$$

$$E[w_i w_j^T] = Q_i \delta_{i,j}, E[v_i v_j^T] = R_i \delta_{i,j}, E[v_i w_j^T] = 0$$

$$Q = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}, R = 0.25$$

where, the input $u(k) \in \mathbb{R}$ is taken as an uncorrelated persistent excitation signal sequence with zero mean and unit variance, $v(k) \in \mathbb{R}$ and $w(k) \in \mathbb{R}^{2 \times 1}$ are white noise sequences with zero mean and variance $\sigma^2$.

Matrices’ coefficients of the augmented system will be
$$F = \begin{bmatrix} 1.00 & 0.00 & 0.00 \\ -1.20 & -0.60 & -0.30 & 0.40 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, G = \begin{bmatrix} 0.20 \\ 0 \end{bmatrix}$$

$$H = \begin{bmatrix} 1.20 & 1.75 \end{bmatrix}$$

Measurement noise covariance doesn’t change but process noise covariance matrix will be changed to

$$Q = \begin{bmatrix} \sigma^2 & 0 & 0 & 0 \\ 0 & \sigma^2 & 0 & 0 \\ 0 & 0 & \sigma^2 & 0 \\ 0 & 0 & 0 & \sigma^2 \end{bmatrix}$$

With the following initial state vector and estimation error covariance matrices’
$$\hat{X}_0^* = \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix}, P_0^* = \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

run the simulation for $\sigma^2 = 0, 0.25, 1$.

![Fig. 1: Operation of the Kalman filter, combining Equation (16) to Equation (22).](image-url)
From Fig. 2, Fig. 3 and Fig. 4 it is clear that decreasing in process noise covariance ($\sigma^2$) causes more similarity in the estimated states and the true states. With regard to this subject that the augmented system is used for estimating the states, for comparison between estimated states and true states, only the first and the second rows of the augmented state vector that are the main states (without time-delay) will be used.

On the other hand, we can see unit time-delay in the state vector as follows:
At each time step, the third and fourth rows of the augmented state vector that are with a unit time-delay are very close to the first and the second rows of the augmented state vector at previous time step. And if we don’t have process noise in the Kalman filter algorithm ($\sigma^2 = 0$), then in $k$‘th time step the third and the fourth rows of the augmented state vector will be equal to the first and the second rows at (k-1)’th time step. For example, the augmented state vector for $\sigma^2 = 0$ and $\sigma^2 = 0.5^2$ in 10 times step is presented in Table I.

In Fig. 5, Fig. 6 and Fig. 7 it can be seen that the estimation error covariance decrease over time steps and finally will be in a good region. For $\sigma^2 = 0$ it will be equal to:

$$
P = 1.0e^{-035} \times 
\begin{bmatrix}
0.0684 & -0.0011 & 0.0017 & 0.0766 \\
-0.0011 & 0.0602 & -0.0801 & 0.0294 \\
0.0017 & -0.1066 & 0.1066 & -0.0388 \\
0.0766 & 0.0294 & -0.0388 & 0.1012
\end{bmatrix}
$$

And for $\sigma^2 = 0.5^2$ it will be:

$$
P = 
\begin{bmatrix}
0.2685 & -0.1511 & -0.0161 & 0.0711 \\
-0.1511 & 0.1554 & -0.0146 & -0.0331 \\
-0.0161 & -0.0146 & 0.4276 & -0.0955 \\
0.0711 & -0.0331 & -0.0955 & 0.3714
\end{bmatrix}
$$

5. Conclusion

This paper discusses state estimation problem for a linear system with unit time-delay using Kalman filter. An augmented system was presented for estimating the states. Required matrices for Kalman filter algorithm in a second-order system were calculated and it was shown that the main system with unit time-delay and the augmented system are equivalent. Finally, performance of the proposed algorithm was studied using an illustrative example in which the estimation error covariance had an acceptable value.
Table I: Augmented state vector in 10 time steps (from k = 20 to k = 30), (a): \( \sigma^2 = 0 \), (b): \( \sigma^2 = 0.5^2 \).

<table>
<thead>
<tr>
<th></th>
<th>X1</th>
<th>X2</th>
<th>X3</th>
<th>X4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>X1</td>
<td>-0.817</td>
<td>-0.373</td>
<td>0.048</td>
<td>1.634</td>
</tr>
<tr>
<td>X2</td>
<td>1.312</td>
<td>0.366</td>
<td>0.888</td>
<td>-0.190</td>
</tr>
<tr>
<td>X3</td>
<td>-1.665</td>
<td>-0.817</td>
<td>-0.373</td>
<td>0.048</td>
</tr>
<tr>
<td>X4</td>
<td>-0.166</td>
<td>1.312</td>
<td>0.366</td>
<td>0.888</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>X1</th>
<th>X2</th>
<th>X3</th>
<th>X4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>X1</td>
<td>-2.468</td>
<td>-3.114</td>
<td>-0.852</td>
<td>1.489</td>
</tr>
<tr>
<td>X2</td>
<td>-1.955</td>
<td>2.134</td>
<td>1.912</td>
<td>1.573</td>
</tr>
<tr>
<td>X3</td>
<td>2.650</td>
<td>-2.695</td>
<td>-2.986</td>
<td>1.450</td>
</tr>
<tr>
<td>X4</td>
<td>-3.536</td>
<td>-1.838</td>
<td>2.056</td>
<td>1.935</td>
</tr>
</tbody>
</table>

Fig. 6: trace of estimation error covariance matrix (\( \sigma^2 = 0.25 \)).

Fig. 7: trace of estimation error covariance matrix (\( \sigma^2 = 1 \)).

References


