Some Properties of Zariski Topology on the Spectrum of Prime Fuzzy Submodules

R. Mahjoob, R. Ameri

Mathematics Department, Faculty of Basic Sciences University of Mazandaran

Babolsar-Iran

e-mail: r.mahjoob@umz.ac.ir, ameri@umz.ac.ir

Abstract

Let $R$ be a commutative ring with identity and let $M$ be an $R$-module. We topologize $FSpec(M)$, the collection of all prime fuzzy submodules of $M$, analogous to that for $FSpec(R)$, the spectrum of fuzzy prime ideals of $R$, and investigate the properties of this topological space.

Keywords: prime fuzzy submodules, fuzzy prime spectrum, $L$-top modules, Zariski topology.

1 Introduction

Let $R$ be a commutative ring with identity and $M$ be a unitary $R$-module. The prime spectrum $Spec(R)$ and the topological space obtained by introducing Zariski topology on the set of prime ideals of a commutative ring with identity play an important role in the fields of commutative algebra, algebraic geometry and lattice theory. Also, recently the notion of prime submodules and Zariski topology on $Spec(M)$, the set of all prime submodules of a module $M$ over a commutative ring with identity $R$, are studied by many authors (for example see [11-14]). As it is well known Zadeh introduced the notion of a fuzzy subset $\mu$ of a nonempty set $X$ as a function from $X$ to unit real interval $I = [0,1]$. J. E. Goguen in [5] replaced $I$ by a complete lattice $L$ in the definition of fuzzy sets and introduced the notion of $L$-fuzzy sets. Fuzzy submodules of $M$ over $R$ were first introduced by Negoita and Ralescu [17]. Pan [21] studied fuzzy finitely generated modules and fuzzy quotient modules.

In the last few years a considerable amount of work has been done on fuzzy ideals in general and prime fuzzy ideals in particular, and some interesting topological properties of the spectrum of fuzzy prime ideals of a ring are obtained (see [4, 6-10]).
Let $M$ be an $R$-module. By $N \leq M$ we mean that $N$ is a submodule of $M$. For any $N \leq M$, we denote the annihilator of $M/N$ by $M : N$, i.e. $N : M = \{ r \in R | rM \subseteq N \}$. In particular, $M : (0)$ is called the annihilator of $M$ and is denoted by $Ann(M)$, that is $Ann(M) = \{ r \in R | rM = 0 \}$. A prime submodule (or a $p$-prime submodule) of $M$ is a proper submodule $P$ with $P : M = p$, such that $rm \in P$ for $r \in R$ and $m \in M$, either $m \in P$ or $r \in p$.

The set of all prime submodules of $M$ is called the prime spectrum of $M$ or, simply the spectrum of $M$ and is denoted by $Spec(M)$.

The authors in [1] introduced and studied the notion of prime Fuzzy submodules of a module $M$ over a commutative ring with identity $R$, where $L$ is a complete lattice. The set of all prime $L$-submodules of $M$ will be called the prime $L$-spectrum of $M$ or, simply the $L$-spectrum of $M$ and is denoted by $L-Spec(M)$. In [2] authors defined a topology on $L-Spec(M)$ and studied some properties of this topology. In this paper we get more topological properties of this topological space under special cases.

2 Preliminaries

Throughout of this paper by $R$ we mean a commutative ring with identity, and $M$ is a unital $R$-module. By a fuzzy subset $\mu$ of a non-empty set $X$, we mean a function $\mu$ from $X$ to $[0, 1]$. $F^X$ denotes the set of all fuzzy subsets of $X$. Let $A$ be a subset of $X$ and $y \in [0, 1]$. Define $y_A \in F^X$ as follows:

$$y_A(x) = \begin{cases} y & x \in A; \\ 0 & \text{otherwise} \end{cases}$$

In special case if $A = \{a\}$ we denote $y_{\{a\}}$ by $y_a$, and it is called a fuzzy point of $X$.

For $\mu \in F^X$ and $a \in [0, 1]$, define $\mu_a$ as follows:

$$\mu_a = \{ x \in X | \mu(x) \geq a \},$$

$\mu_a$ is called the $a$-cut or $a$-level subset of $\mu$. The image of $\mu$ is denote by $Im(\mu)$ or $\mu(X)$. For $\mu, \nu \in F^X$ we say that $\mu$ is contained in $\nu$ and we write $\mu \subseteq \nu$ if for all $x \in X$, $\mu(x) \leq \nu(x)$.

For $\mu, \nu \in F^X$, the intersection and union, $\mu \cup \nu$, $\mu \cap \nu \in F^X$, are defined by

$$(\mu \cup \nu)(x) = \mu(x) \lor \nu(x) \quad \text{and} \quad (\mu \cap \nu)(x) = \mu(x) \land \nu(x),$$

for all $x \in X$.

We recall some definitions and theorems from the book [15], which we need for the development of our paper.

**Definition 2.1.** Let $\mu \in F^R$. Then $\mu$ is called a fuzzy ideal of $R$ if for every $x, y \in R$ the following conditions are satisfied:

1) $\mu(x - y) \geq \mu(x) \land \mu(y)$;
For $\mu(x y) \geq \mu(x) \lor \mu(y)$.

The set of all fuzzy ideals of $R$ is denoted by $FI(R)$. For $\mu, \nu \in FI(R)$, the product of $\mu$ and $\nu$ is defined by

$\mu \nu(x) = \bigvee \{\mu(y) \land \nu(z)|y, z \in R, x = yz\} \forall x \in R,$

and in [15] it was proved that $\mu \nu \in FI(R)$.

**Definition 2.2.** $\zeta \in FI(R)$ is called a prime fuzzy ideal of $R$ if $\zeta$ is non-constant and for every $\mu, \nu \in FI(R)$, $\mu \nu \subseteq \zeta$ implies that $\mu \subseteq \zeta$ or $\nu \subseteq \zeta$.

By $FSpec(R)$ we mean the set of all prime fuzzy ideals of $R$.

**Definition 2.3.** Let $\zeta \in F^R$ and $\mu \in F^M$. Define $\zeta \cdot \mu \in F^M$ as follows:

$$(\zeta \cdot \mu)(x) = \bigvee \{\zeta(r) \land \mu(y)|r \in R, y \in M, ry = x\} \text{ for all } x \in M.$$ 

**Definition 2.4.** A fuzzy subset $\mu \in F^M$ is a fuzzy submodule of $M$ if:

1. $\mu(0) = 1$;
2. $\mu(r x) \geq \mu(x)$ for all $r \in R$ and $x \in M$;
3. $\mu(x + y) \geq \mu(x) \land \mu(y)$ for all $x, y \in M$.

The set of all fuzzy submodules of $M$ is denoted by $F(M)$.

For $\mu, \nu \in F^M$ and $\zeta \in F^R$, $\mu : \nu \in F^R$ and $\mu : \zeta \in F^M$ are defined as follows:

$$\mu : \nu = \bigcup \{\eta|\eta \in F^R, \eta \cdot \nu \subseteq \mu\}$$
$$\mu : \zeta = \bigcup \{\nu|\nu \in F^M, \zeta \cdot \nu \subseteq \mu\}.$$ 

In [15] it was proved that if $\nu \in F^M$, $\mu \in F(M)$, and $\zeta \in FI(R)$, then $\mu : \nu = \bigcup \{\eta|\eta \in FI(R), \eta \cdot \nu \subseteq \mu\}$ and $\mu : \zeta = \bigcup \{\nu|\nu \in F(M), \zeta \cdot \nu \subseteq \mu\}$. Also, it was proved that if $\mu \in F(M), \nu \in F^M, \zeta \in FI(R)$, then $\mu : \nu \in FI(R)$ and $\mu : \zeta \in F(M)$.

**Definition 2.5[1].** A non-constant fuzzy submodule $\mu$ of $M$ is said to be prime if for $\zeta \in FI(R)$ and $\nu \in F(M)$ such that $\zeta \cdot \nu \subseteq \mu$ then either $\nu \subseteq \mu$ or $\zeta \subseteq \mu : 1_M$.

In the sequel $FSpec(M)$ denotes the set of all prime fuzzy submodules of $M$.

In [1] we obtained the following results:

**Theorem 2.6 [1].** $\mu \in FSpec(M)$ if and only if $\mu = 1_M \cup c_M$ such that $\mu_s = \{x \in M|\mu(x) = 1\}$ is a prime submodule of $M$ and $c \in [0, 1]$.

**Corollary 2.7 [1].** If $\mu \in FSpec(M)$, then $\mu : 1_M$ is a prime fuzzy ideal of $R$.

For $\mu \in F^M$ set $V^*(\mu) = \{P \in FSpec(M)|\mu \subseteq P\}$ and $V(\mu) = \{P \in FSpec(M)|\mu : 1_M \subseteq P : 1_M\}$.

Now, we put

$$F_\zeta^*(M) = \{V^*(\mu)|\mu \in F(M)\};$$
Proposition 2.8. Let \( X \) be a prime fuzzy submodule of \( R \). Then by Corollary 2.7 we have \( \mu : 1_M \) is prime fuzzy ideal of \( R \). Define \( \mu : 1_M \in FR/Ann(M) \) as follows:

\[
\overline{\mu : 1_M}([x]) = \bigvee \{\mu : 1_M(z) | z \in [x]\}
\]

and the map \( \psi : FSpec(M) \rightarrow FSpec(R/Ann(M)) \) by

\[
\psi(\mu) = \overline{\mu : 1_M} \quad \text{for} \quad \mu \in FSpec(M),
\]

\( \psi \) is called the natural map.

In the sequel by \( X \) we mean \( FSpec(M) \) and by \( \overline{X} \) we mean \( FSpec(R/Ann(M)) \).

For any \( R \)-module \( M \), we consider the set \( B = \{D(x, 1_M) | x \in R, \beta \in (0, 1]\} \) such that \( D(x, 1_M) = X \setminus V(x, 1_M) \). In [2] it is proved that \( B \) forms a base for Zariski topology on \( X = FSpec(M) \).

Proposition 2.8. Let \( M \) be an \( R \)-module. If the natural map \( \psi \) is surjective, then \( X \) is compact.

Proof. Let \( X = \bigcup \{D(x, 1_M) | x \in R, \beta \in (0, 1]\} \). Then

\[
\overline{X} = \psi(X) = \psi(\bigcup \{D(x, 1_M) | x \in R, \beta \in (0, 1]\}) = \bigcup \{\psi(D(x, 1_M)) | x \in R, \beta \in (0, 1]\} = \bigcup \{\overline{x_\beta} | x \in R, \beta \in (0, 1]\} \quad \text{(since \( \psi \) is surjective)}.
\]
Also, since $X$ is compact, we can write $X = \bigcup_{i=1}^{n} (x_i)_{\beta_i}$, and hence $\psi^{-1}(X) = \psi^{-1}\left(\bigcup_{i=1}^{n} (x_i)_{\beta_i}\right)$. Thus $X = \bigcup_{i=1}^{n} \psi^{-1}\left((x_i)_{\beta_i}\right)$, and so $\psi^{-1}\left((x_i)_{\beta_i}\right) = ((x_i)_{\beta_i},1_M)$. Therefore is $X$ is compact.

### 3 Main Results

Let $M$ be an $R$-module. For $Y \subseteq X$ we denote the intersection of all elements in $Y$ by $\Gamma(Y)$ and closure of $Y$ in $X$ for this topology by $\overline{Y}$.

**Proposition 3.1.** Let $M$ be an $R$-module and $Y \subseteq X$. Then $V(\Gamma(Y)) = \overline{Y}$. Hence $Y$ is closed if and only if $V(\Gamma(Y)) = Y$

**Proof.** Let $P \in Y$, then $\Gamma(Y) \subseteq P$, so $P \in V(\Gamma(Y))$ and then $Y \subseteq V(\Gamma(Y))$. Let $V(\mu)$ be any closed subset of $X$ such that $Y \subseteq V(\mu)$. Then for every $P \in Y, P \in V(\mu)$, so

$$
\mu : 1_M \subseteq P : 1_M \implies \mu : 1_M \subseteq \bigcap_{P \in Y} (P : 1_M) = (\bigcap_{P \in Y} P) : 1_M = \Gamma(Y) : 1_M
$$

Now let $Q \in V(\Gamma(Y))$, then $\Gamma(Y) : 1_M \subseteq Q : 1_M$, but $\mu : 1_M \subseteq \Gamma(Y) : 1_M \subseteq Q : 1_M$, so $Q \in V(\mu)$. Then we conclude that $V(\Gamma(Y)) \subseteq V(\mu)$ and then $\overline{Y} = V(\Gamma(Y))$.

By this result it is easy to see that $Y$ is closed subset if and only if $V(\Gamma(Y)) = Y$.

**Lemma 3.2.** Every fuzzy ideal of $R$ is contained in a maximal fuzzy ideal.

**Proposition 3.3.** Let $\nu$ be a maximal fuzzy ideal of $R$. Then $\nu.1_M$ is a prime fuzzy submodule of $M$.

**Proof.** Let $\nu$ be maximal, then $\nu = 1_{\nu} + cR$ such that $\nu_{s}$ is a maximal ideal of $R$ and $c \in [0,1)$. We have $\nu.1_M = 1_{\nu.1_M} + cM$. But since $\nu_{s}$ is maximal then $\nu_{s}M$ is a prime submodule of $M$ [See 10]. Therefore by Theorem 2.6, $\nu.1_M$ is a prime fuzzy submodule of $M$.

**Definition 3.4.** $\mu \in F(M)$ is called maximal prime fuzzy submodule if $\mu \in FSpec(M)$ and there is not any $\nu \in FSpec(M)$ such that $\mu \not\subseteq \nu$.

**Lemma 3.5.** If $\mu \in FSpec(M)$ is maximal prime then $\mu : 1_M$ is a maximal fuzzy ideal of $R$.

**Proof.** Let $\mu \in FSpec(M)$ be a maximal prime and let for $\eta \in FI(R), \mu : 1_M \subseteq \eta$. (1). By Lemma 3.2, there exists a maximal fuzzy ideal $m$ of $R$ such that $\eta \subseteq m$. Since $\mu : 1_M \subseteq \eta$, then $\mu \subseteq \eta.1_M \subseteq m.1_M$ and by Proposition 3.3, $m.1_M$ is a fuzzy prime submodule of $M$. Since $\mu$ is maximal prime then $m = 1_M$ and so $\mu = \eta.1_M$ and hence $\eta \subseteq \mu : 1_M$ (2). By (1), (2) we have $\mu : 1_M = \eta$ and thus $\mu : 1_M$ is a maximal fuzzy ideal of $R$. 

**هفتامین کنفرانس مسئله‌های فازی ۹- V. شهریور ۱۳۸۷. اسلام‌آباد، فارسی مشهد**
**Proposition 3.6.** For an $R$-module $M$, let $P$ be any element of $X$ Then

1) $\{P\} = V(P)$

2) For any $Q \in X, Q \in \{P\}$ if and only if $P : 1_M \subseteq Q : 1_M$ if and only if $V(Q) \subseteq V(P)$.

3) The set $\{P\}$ is closed if and only if
   
   a) $P$ is a maximal prime fuzzy submodule of $M$.
   
   b) $FSpec_p(M) = \{P\}$ such that $P : 1_M = p$.

The previous Proposition establishes the conditions such that $X$ is $T_1$. It says that $X$ is $T_1$ space if and only if every prime fuzzy submodule of $M$ is maximal prime and $|FSpec_p(M)| \leq 1$ for all $p \in FSpec(R)$.

**Definition 3.7.** A topological space $A$ is called irreducible if for any decomposition $A = A_1 \cup A_2$ with closed subsets $A_1$ and $A_2$ of $A$ we have $A_1 = A$ or $A_2 = A$. A subspace $A'$ of $A$ is irreducible if it is irreducible as a subspace of $A$.

**Theorem 3.8.** $V(P)$ is an irreducible closed subset of $FSpec(M)$ for every prime fuzzy submodule $P$ of $M$.

**Corollary 3.9.** Let $Y$ be a subset of $FSpec(M)$. If $\Gamma(Y)$ is a prime fuzzy submodule of $M$, then $Y$ is irreducible.

**Corollary 3.10.** Let $P^* = \bigcap_{P \in X} P$. If $P^*$ is a prime fuzzy submodule of $M$ then $X$ is irreducible.

**Corollary 3.11.** Let $M$ be an $R$-module:

1) Let $Y = \{P_i \in X | i \in I\}$ be which is linearly ordered by inclusion, then $Y$ is irreducible in $X$.

2) $FSpec_p(M)$ is irreducible for $p \in FSpec(R)$

3) if $p$ is a maximal fuzzy ideal of $R$ then $Fspec_p(M)$ is an irreducible closed subset of $X$.

**Proof.** 1) Since elements of $Y$ is linearly ordered by inclusion, then $\Gamma(Y)$ is a prime fuzzy submodule of $M$. Then by Corollary 3.9, $Y$ is irreducible.

2) We show that $\Gamma(FSpec_p(M))$ is a prime fuzzy submodule of $M$. We have $\bigcap_{P \in FSpec_p(M)} 1_M = \bigcap_{P \in FSpec_p(M)} (P : 1_M) = p$. So

$\Gamma(FSpec_p(M)) : 1_M = p$. Now let $\alpha \in FI(R)$ and $\nu \in F(M)$ and $\alpha \nu \subseteq \Gamma(FSpec_p(M))$ and $\nu \not\subseteq \Gamma(FSpec_p(M))$, then there exists $P' \in FSpec_p(M)$ such that $\nu \not\subseteq P'$. Therefore $\alpha \subseteq P' : 1_M = p = \Gamma(FSpec_p(M)) : 1_M$. This means that $\Gamma(FSpec_p(M))$ is prime fuzzy submodule and by Corollary 3.9 $FSpec_p(M)$ is irreducible.

3) let $p$ be a maximal fuzzy ideal of $R$. By (2), $FSpec_p(M)$ is irreducible. But since $p$ is maximal then $(p.1_M) : 1_M = p$. Now let $P \in V(p.1_M)$ then $p = (p.1_M) : 1_M \subseteq P : 1_M$ and since $p$ is maximal then

$P : 1_M = p \implies P \in FSpec_p(M) \implies V(p.1_M) \subseteq FSpec_p(M)$

(1)

But for $P \in FSpec_p(M), P : 1_M = p = (p.1_M) : 1_M$, so

$P \in V(p.1_M) \implies Fspec_p(M) \subseteq V(p.1_M)$

(2)
From (1), (2) we have $V(p.1_M) = Fspec_p(M)$ and by this $Fspec_p(M)$ is closed.

**Corollary 3.12.** Let $Y$ be a subset of $X$ and let $\Gamma(Y) : 1_M = p$ is a prime fuzzy ideal of $R$. If $Fspec_p(M) \neq \emptyset$ then $Y$ is irreducible.

**Theorem 3.13.** For the topological space $X$ the following statements are equivalent:

i) $X$ is $T_0$ space.

ii) the natural map $\psi$ is injective.

iii) if $V(P) = V(Q)$ then $P = Q$ for any $P, Q \in X$

iv) $|FSpec_p(M)| \leq 1$ for every $p \in FSpec(R)$

**Corollary 3.14.** If $M$ is a fuzzy top module, then $Fspec(M)$ is a $T_0$ space for Zariski topology and $\tau^*$.

**Proof.** Let $P, Q \in X$ and $P \neq Q$. Then $P \nsubseteq Q$ or $Q \nsubseteq P$. Let $P \nsubseteq Q$, then $Q \notin V^*(P)$, so $Q \in D^*(P)$, but $P \notin D^*(Q)$ and $D^*(P)$ is an open set in topology $\tau^*$. This shows that $X$ is $T_0$ for topology $\tau^*$. Since $\tau \leq \tau^*$ then $Fspec(M)$ is $T_0$ for Zariski topology too.

Let $n = \{P : 1_M|P \in FSpec(M)\}$ and $n^* = \{p_*|p \in n\}$.

**Lemma 3.15.** $D(x_\beta.1_M) = \emptyset$ if and only if $x \in \bigcap n^*$.

**Proof.** Let $D(x_\beta.1_M) = \emptyset$, then $V(x_\beta.1_M) = X$. Let $P$ be a prime submodule of $M$ and put $\mu = \chi_p$, then $\mu \in FSpec(M)$. Let $p = \mu : 1_M$. Then $(x_\beta.1_M) : 1_M \subseteq \mu : 1_M = p$, but $x_\beta \subseteq (x_\beta.1_M) : 1_M$, therefore $x_\beta \subseteq p$, thus $\beta \leq p(x) = 1$. This shows that $x \in P_*$ and then $x \in \bigcap n^*$. Conversely suppose that $x \in \bigcap n^*$ and let $P \in Fspec(M)$. If $p = P : 1_M$ then $x \in p_*$, so

$$p(x) = 1 \implies (P : 1_M)(x) = 1 \implies x_\beta \subseteq P : 1_M \implies x_\beta.1_M \subseteq P \implies (x_\beta.1_M) : 1_M \subseteq P : 1_M$$

therefore $P \in V(x_\beta.1_M)$ and then $V(x_\beta.1_M) = X$. Thus $D(x_\beta.1_M) = \emptyset$.

**Corollary 3.16.** If $x$ is nilpotent then $D(x_\beta.1_M) = \emptyset$ for $\beta \in (0, 1]$.

Let $X = FSpec(M)$ for an $R-$module $M$ and let $\alpha \in [0, 1)$. We shall denote the subspace $\{\mu \in X|Im\mu = \{1, \alpha\}\}$ by $A_\alpha$.

**Lemma 3.17.** If the natural map $\psi$ is injective and every prime ideal of $R$ is maximal then the subspace $A_\alpha$ of $X$ is Hausdorff.

**REFERENCES**


