MULTI-OBJECTIVE SEMIDEFINITE PROGRAMMING

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ABSTRACT

In this paper we recall multi-objective programming and semidefinite programming problems, then we introduce multi-objective semidefinite programming problem and prove some theorems about it.

Keywords: Multi-objective; Semidefinite programming; Efficiency.

1. INTRODUCTION

In this section we recall multi-objective programming and semidefinite programming problem.

1.1. Multi-objective programming problem

Multi-objective optimization is an area of multiple criteria decision making. A multi-objective problem is an optimization problem that has multiple objective functions, and can be formulated as follows:

\[
\min \quad f_1(x), f_2(x), \ldots, f_k(x) \\
\text{s.t.} \quad g_i(x) \leq b_i, \quad i = 1, 2, \ldots, m
\]

(1.1)

Ab obviously if k=1, problem 1.1 is an ordinary optimization problem. We recall some definitions and some theorems about multi-objective problem.

Definition 1.1. A feasible point x is called efficient if there exist no feasible point y such that \( f_i(x) \geq f_i(y) \) for all \( i = 1, 2, \ldots, k \), for some \( 1 \leq j \leq k \), \( f_j(x) > f_j(y) \).

Definition 1.2. A feasible point x is called weak efficient if there exist no feasible point y such that \( f_i(x) > f_i(y) \) for all \( i = 1, 2, \ldots, k \).

If objective functions and constraints are linear, problem 1.1 is called multi-objective linear programming (MOLP). There is some approaches to solve an MOLP problem, one of them is the weighted sum method.

Theorem 1.3. A feasible point \( x^* \) is efficient if and only if there exist \( \lambda_i > 0 \), \( i = 1, 2, \ldots, k \) such that x is optimal for the following problem

\[
\min \quad \sum_{i=1}^{k} \lambda_i f_i(x) \\
\text{s.t.} \quad g_i(x) \leq b_i, \quad i = 1, 2, \ldots, m
\]

(1.2)

1.2. Semidefinite programming

Semidefinite programming is a main branch of convex optimization which introduced in 1990’ s. The general form of this kind of problem is as follows:

\[
\min \quad C \bullet X \\
\text{s.t.} \quad A_i \bullet X = b_i, \quad i = 1, 2, \ldots, m \\
X \succeq 0.
\]

(1.3)

where \( C \), \( A_i \) and \( X \) are n-by-n symmetric matrices and \( b_i \in \mathbb{R} \). The notation \( \succeq \) shows the matrix \( X \) must be semidefinite matrix, and the notation \( \bullet \) is inner product in matrix space. Many practical problems in combinatorial optimization can be modeled or approximated as semidefinite programming problems such as max cut problem. The dual of semidefinite programming can be defined as follows:

\[
\max \quad \sum_{i=1}^{m} b_i y_i \\
\text{s.t.} \quad \sum_{i=1}^{m} y_i A_i \preceq C.
\]

(1.4)

Analogously to linear programming, we have weak duality and strong duality theorems.

Theorem 1.4. The value of objective function problem 1.4 for any feasible is a lower-bounds for the value of objective function problem 1.3 for any feasible solution. Also The value of objective function problem 1.3 for any feasible is a upper-bounds for the value of objective function problem 1.4 for any feasible solution.

For proof see [1].

Theorem 1.5. Under well known condition if \( X^* \) is an optimal point for primal and \( y^* = (y_1^*, y_2^*, \ldots, y_m^*) \) is an optimal point for dual we have

\[
C \bullet X^* = \sum_{i=1}^{m} b_i y_i^*.
\]

And also if \( X^* \) and \( y^* \) are feasible point and equality \( C \bullet X^* = \sum_{i=1}^{m} b_i y_i^* \) is hold then \( X^* \) and \( y^* \) are optimal.

For proof see [1].

2. MULTI-OBJECTIVE SEMIDEFINITE PROGRAMMING

In this section we introduce multi-objective semidefinite programming and present some theorems to find efficient point.

A multi-objective semidefinite programming problem is a multi-objective problem as follows:

\[
\min \quad C_1 \bullet X, C_2 \bullet X, \ldots, C_k \bullet X \\
\text{s.t.} \quad A_i \bullet X = b_i, \quad i = 1, 2, \ldots, m
\]

(2.1)

\[
X \succeq 0.
\]
Theorem 2.1. Suppose that \( \lambda_i > 0, i = 1, 2, ..., k \) and consider following problem

\[
\min \sum_{i=1}^{k} \lambda_i C_i \cdot X \\
\text{s.t.} \quad A_i \cdot X = b_i, \quad i = 1, 2, ..., m \\
X \geq 0.
\]

Let \( X^* \) be an optimal solution of problem 2.3 then \( X^* \) is an efficient point for problem 2.1.

Proof. Suppose that \( X^* \) is not efficient, then there exist a feasible point \( Y \) such that \( \lambda_i C_i \cdot Y \leq \lambda_i C_i \cdot X^* \) and for some \( j \), \( C_j \cdot Y < C_j \cdot X^* \), so \( \lambda_i C_i \cdot Y \leq \lambda_i C_i \cdot X^* \) and also \( \lambda_i C_j \cdot Y < \lambda_i C_j \cdot X^* \) therefor \( \sum_{i=1}^{k} \lambda_i C_i \cdot Y < \sum_{i=1}^{k} \lambda_i C_i \cdot X^* \), it is contradic to optimality of \( X^* \).

Theorem 2.2. Under well known condition if \( X^* \) be an efficient point then there exist \( \lambda_i > 0, i = 1, 2, ..., k \) such that \( X^* \) is an optimal point for the following problem

\[
\min \sum_{i=1}^{k} \lambda_i C_i \cdot X \\
\text{s.t.} \quad A_i \cdot X = b_i, \quad i = 1, 2, ..., m \\
X \geq 0.
\]

Proof. Suppose that \( X^* \) is an efficient point, consider the following problem:

\[
\min \sum_{j=1}^{k} C_j \cdot X \\
\text{s.t.} \quad A_i \cdot X = b_i, \quad i = 1, 2, ..., m \\
C_j \cdot X \leq C_j \cdot X^*, j = 1, 2, ..., k \\
X \geq 0.
\]

Suppose that \( X^* \) is not optimal for problem 2.4 and let \( X^{**} \) be an optimal point then we have \( \sum_{j=1}^{k} C_j \cdot X^{**} < \sum_{j=1}^{k} C_j \cdot X^* \) and also we have \( C_j \cdot X^{**} \leq C_j \cdot X^* \cdot X^* \). So \( C_j \cdot X^{**} < C_j \cdot X^* \) for some \( 1 \leq l \leq k \). So \( X^* \) is not an efficient point, the contradiction implies \( X^* \) be optimal.

Now consider the dual of problem 2.4:

\[
\min \sum_{j=1}^{k} C_j \cdot X^* y_j + \sum_{i=1}^{m} w_i b_i \\
\text{s.t.} \quad \sum_{j=1}^{k} w_j A_j + \sum_{j=1}^{k} C_j y_j \leq \sum_{j=1}^{k} C_j \\
y_j \leq 0.
\]

From the strong duality theorem, we have

\[
\sum_{j=1}^{k} C_j \cdot X^* y_j + \sum_{i=1}^{m} w_i b_i = \sum_{j=1}^{k} C_j \cdot X^* 
\]

so

\[
\sum_{i=1}^{m} w_i^* b_i = \sum_{j=1}^{k} C_j \cdot X^* - \sum_{j=1}^{k} C_j \cdot X^* y_j = \sum_{j=1}^{k} C_j \cdot X^*(1 - y_j^*).
\]

Note that \( 1 - y_j^* > 0 \). Now set \( \lambda_j = 1 - y_j^* > 0 \) and consider the following problem

\[
\min \sum_{j=1}^{k} (1 - y_j^*) C_j \cdot X \\
\text{s.t.} \quad A_i \cdot X = b_i, \quad i = 1, 2, ..., m \\
X \geq 0.
\]

and the dual of problem 2.6 is as follows:

\[
\min \sum_{i=1}^{m} w_i^* b_i \\
\text{s.t.} \quad \sum_{j=1}^{m} w_i A_i \geq \sum_{j=1}^{k} (1 + y_j^*) C_j \\
h_j \geq 0, X \geq 0.
\]

Abivously \( X^* \) and \( w^* \) are feasible and on the other hand the value of objective functions problem 2.7 and problem 2.6 in \( X^* \) and \( w^* \) are equal. So from the duality theorem we have \( X^* \) and \( w^* \) are optimal.

If we have a feasible point we can find an efficient point by solving the following problem:

\[
\max \sum_{j=1}^{k} h_j \\
\text{s.t.} \quad A_i \cdot X = b_i, \quad i = 1, 2, ..., m \\
C_j \cdot X + h_j = C_j \cdot X^*, j = 1, 2, ..., k \\
h_j \geq 0, X \geq 0.
\]

where \( X \) is a feasible point and any optimal solution is efficient. We prove this statement.

Theorem 2.3. If \( X^* \) is a feasible point for problem 2.1 then any optimal solution of problem 2.9 is an efficient point.

Proof. Suppose that \( X^* \) is an optimal solution of problem 2.9 and \( X^* \) is not efficient. So there exist some feasible point such as \( X'' \) which \( C_j \cdot X'' \leq C_j \cdot X^* \) and for some \( 1 \leq l \leq k \), \( C_j \cdot X'' < C_j \cdot X^* \) hence we can reduce the objective value of problem 2.9.

3. REFERENCES