SIMPLEX METHOD FOR MULTIPLE OBJECTIVE LINEAR PROGRAMS WITH BOUNDED VARIABLES

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ABSTRACT

We develop a multicriteria simplex method for multiple objective linear programs with bounded variables and compute all nondominated basic feasible solutions (extreme points). The approach is based on the usual multicriteria simplex method and the extended simplex method for single objective linear programs with bounded variables.

Keywords: multiple objective linear programming; bounded variables; extended simplex algorithm; nondominated basic feasible solutions.

1. INTRODUCTION

consider the multiple objective linear program (MOLP) with upper bounds in the standard form,

\[
\begin{align*}
\text{max } & \mathbf{C} \mathbf{x} \\
\text{s.t. } & \mathbf{A} \mathbf{x} = \mathbf{b} \\
& 0 \leq \mathbf{x} \leq \mathbf{b},
\end{align*}
\]

where \( \mathbf{C} \) is a real \( p \times n \) matrix consisting of the rows \( x^T \), \( k = 1, \ldots, p \), \( \mathbf{A} \) is a real \( m \times n \) matrix, \( \mathbf{b} \) is a real \( m \times 1 \) vector and \( \mathbf{b} \) is a real \( n \times 1 \) vector. Without loss of generality, we assume that rank of \( \mathbf{A} \) is \( m \).

The bounded-variable problem (1.1) can be solved using the usual multicriteria simplex method [1, 3, 4], by adding slack variables to the upper-bound constraints, thereby converting them to equalities. However, this approach incurs high computing and storage costs. Here, instead, we are to extend the usual simplex method for multiple objective linear programs introduced by Zeleny [4] to consider the bounded-variable constraints implicitly.

Throughout, \( \mathbb{R}^n \) denotes the \( n \)-dimensional Euclidean space. If \( \mathbf{v} \in \mathbb{R}^n \) and \( \mathbf{A} \in \mathbb{R}^{m,n} \), then by \( \mathbf{v} \) we denote the \( i \)-th component of \( \mathbf{v} \) and by \( \mathbf{v} \) we denote the \( j \)-th column of \( \mathbf{A} \). For two vectors \( \mathbf{a} \) and \( \mathbf{b} \) in \( \mathbb{R}^n \), by \( \mathbf{a} \geq \mathbf{b} \), we mean \( a_i \geq b_i \), for all \( i = 1, \ldots, n \), and \( \mathbf{a} > \mathbf{b} \), we mean \( a_i > b_i \), for all \( i = 1, \ldots, n \), and \( \mathbf{a} > \mathbf{b} \), for at least one \( j \in \{1, \ldots, n\} \), and by \( \mathbf{a} > \mathbf{b} \), we mean \( a_i > b_i \), for all \( i = 1, \ldots, n \).

For program (1.1) we have some notations and definitions as follows. The feasible set is \( \mathcal{X} = \{ \mathbf{x} \in \mathbb{R}^n : \mathbf{A} \mathbf{x} = \mathbf{b}, \ 0 \leq \mathbf{x} \leq \mathbf{b} \} \). Given \( \mathbf{x}^1, \mathbf{x}^2 \in \mathcal{X} \), we say that \( \mathbf{x}^1 \) is dominated by \( \mathbf{x}^2 \) if \( \mathbf{C} \mathbf{x}^1 \leq \mathbf{C} \mathbf{x}^2 \). A point \( \mathbf{x} \in \mathcal{X} \) is an efficient (nondominated) solution if it is not dominated by any other feasible point. Consider \( \mathcal{B} \subseteq \{1, \ldots, n\} \) be a set of indices that corresponds to \( m \) linearly independent column of \( \mathbf{A} \), and let \( \mathcal{N} = \{1, \ldots, n\} \setminus \mathcal{B} \). Consider a rearrangement of \( \mathbf{A} \) as \( [\mathbf{B} \mathbf{N}] \), where \( \mathbf{B} \) is the \( m \times m \) matrix whose columns are indexed by the indices in \( \mathcal{B} \), and \( \mathbf{N} \) is the \( m \times (n - m) \) matrix whose columns are indexed by the indices in \( \mathcal{N} \). Similarly, consider \( \mathbf{x} \) as \( \begin{bmatrix} \mathbf{x}^B, & \mathbf{x}^N \end{bmatrix}^T \). The components of \( \mathbf{x}^B \) and \( \mathbf{x}^N \) are called basic and nonbasic variables respectively. An extended basic feasible solution corresponding to program (1.1) is a feasible solution for which \( n - m \) (nonbasic) variables are equal to either their lower bounds (zero) or their upper bounds, and the remaining \( m \) (basic) variables correspond to linearly independent columns of \( \mathbf{A} \).

2. SIMPLEX METHOD FOR MOLP WITH BOUNDED VARIABLES

In this section we modify the usual multicriteria simplex method introduced by Zeleny [4], for multiple objective linear program with bounded variables, problem (1.1), to compute all nondominated basic feasible solutions (extreme points). Each iterate generated by the extended multicriteria simplex method is an extended basic feasible solution. A general extended multicriteria simplex tableau for problem (1.1) is shown in Table 1, expressing the following relations:

\[ \mathbf{X}_B + \mathbf{Y}_X = \mathbf{y}_0, \]

\[ \mathbf{e}^T = \mathbf{y}_0 - \mathbf{y}^B, \]

where \( \mathbf{y}^B = \mathbf{C}^T \mathbf{B}^{-1} \mathbf{N} - \mathbf{C}^T \mathbf{X} \) and \( \mathbf{y}_0 = \mathbf{C}^T \mathbf{B}^{-1} \mathbf{b} \), for \( k = 1, \ldots, p \).

Each row, \( \mathbf{x}_k \), \( i \in \mathcal{N} \), is equal to zero or \( \mathbf{h}_i \), with respect to the definition of the extended basic feasible solution. Here, we are assuming that every extended basic feasible solution is nondegenerate, which means that the \( m \) basic variables take values not equal to either of their bounds.

<table>
<thead>
<tr>
<th>Reduced cost rows</th>
<th>( \mathbf{e}_B )</th>
<th>( \mathbf{e}_N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{X}_B )</td>
<td>( 0_{m \times p} )</td>
<td>( \mathbf{I}_m )</td>
</tr>
</tbody>
</table>

In the extended multicriteria simplex method, whenever \( \mathbf{x}_i \) is nonbasic at its upper bound \( \mathbf{h}_i \), we substitute \( \mathbf{h}_i - \mathbf{x}_i \) for \( \mathbf{x}_i \) in the model making \( \mathbf{x}_i \) nonbasic at value zero. For this, it is convenient to introduce the notations \( \mathbf{x}^+_i = \mathbf{x}_i + \mathbf{e}_i \) and \( \mathbf{x}^-_i = \mathbf{h}_i - \mathbf{x}_i \). In the indicator row of the extended multicriteria simplex tableau, for the \( i \)-th column corresponding to \( \mathbf{x}^+_i = \mathbf{x}_i \), an indicator of \( \mathbf{e}_i = + \), and for \( \mathbf{x}^-_i = \mathbf{h}_i - \mathbf{x}_i \), an indicator of \( \mathbf{e}_i = - \), are specified.
Remark 2.1. Everywhere in this section, by $\theta_j$, we mean the minimum of three numbers $h_j$, $t_1 = \min \left( \frac{y_{ij}}{y_{ij}^j}; y_{ij} > 0 \right)$ and $t_2 = \min \left( \frac{x_{ij} - h_j}{y_{ij}^j}; y_{ij} < 0 \right) (h_j$ is the upper bound associated with basic variable $x_j$); and by introducing the nonbasic variable $x_j$, we mean that the extended multicriteria simplex tableau is updated as follows (for more details, see Section 3.6 in [2]):

(a) If $\theta_j = h_j$ then the variable $x_j$ goes to its opposite bound, that is, substitute $h_j - x_j$ for $x_j$. To do this, subtract $h_j$ times column $j$ from column 0. Multiply column $j$ by minus unity (including a change in sign of $e_j$). The basis does not change and no pivot is required.

(b) If $\theta_j = t_1$ then suppose $i$ is the minimizing index in $t_1$. Then, the $i$th basic variable returns to its old bound. Pivot on the $(i,j)$th element.

(c) If $\theta_j = t_2$ then suppose $i$ is the minimizing index in $t_2$. Then, the $i$th basic variable goes to its opposite bound. Substitute $h_j - x_j$ for $x_j$ (subtract $h_j$ from $y_{0j}$ and change the signs of $y_{ij}$ and $e_i$), and pivot on the $(i,j)$th element.

Step 1: Start with an extended basic feasible solution $x^0$ and form the initial extended multicriteria simplex tableau.

Step 2: Check if the current solution is efficient or not.

(i) Use Remark 2.5 to check whether $x^0$ is efficient. If $x^0$ is efficient then go to Step 3.

(ii) Use Theorem 2.2 to check whether $x^0$ is not efficient. If $x^0$ is not efficient then go to Step 3.

(iii) Apply Zeleny’s algorithm given in [4]. Find the solution $v^*$ of the following linear program:

$$\begin{align*}
\text{max } v & = \sum_{k=1}^{p} \delta_k \\
\text{s.t. } & \quad Ax = b \\
& \quad c^k x - \delta_k = c^k x^0, \quad k = 1, \ldots, p \\
& \quad 0 \leq x \leq h \\
& \quad \delta_k \geq 0, \quad k = 1, \ldots, p.
\end{align*}$$

If $v^* = 0$ then $x^0$ is an efficient solution.

Step 3: (i) Find all nonbasic variables $x_j$ so that the conditions of Theorem 2.2 are established and push all such nonbasic variables and the corresponding tableau one by one onto the stack and go to Step 4.

(ii) If there is a nonbasic variable $x_j$ for which all $q \in N \setminus \{j\}, x_j$ and $x_k$ satisfy the conditions of Theorem 2.4 then push $x_j$ and the tableau onto the stack and go to Step 4.

(iii) Find all nonbasic variables $x_j$ with $y_{0j} = (y_{01,j}, \ldots, y_{0p,j})^T$ having both positive and negative entries and push all such nonbasic variables and the corresponding tableau one by one onto the stack and go to Step 4.

Step 4: If the stack is not empty then pop off the stack to get $x_j$ and the corresponding tableau. Introduce $x_j$ and update the extended multicriteria simplex tableau by Remark 2.1 to get the new solution $x^1$. If $x^1$ corresponds to a tableau met before then go to Step 4 else set $x^0 = x^1$ and go to Step 2.

Algorithm 1: Extended simplex algorithm for MOLP with bounded variables (ESA-MOLPBV)

3. REFERENCES


