EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, using the fixed point index theory and properties of Green function we study the existence and multiplicity of positive solutions of nonlinear fractional differential equations and obtain some new results.

1. Introduction

In this paper, we consider the following initial value problem

\[ D^\beta u(t) = f(t, D^\alpha u(t)), \quad t \in (0, 1), \]
\[ u^{(k)}(0) = \eta_k, \quad k = 0, 1, ..., m - 1, \]

where \( n - 1 < \beta < \alpha < n \), \( n \in \mathbb{N} \), \( D^\alpha, D^\beta \) are the Caputo fractional derivatives and \( f \in C([0, 1] \times \mathbb{R}) \). In [3, 4, 5], the authors considered the existence of solutions of the initial value problem (1.1)-(1.2). They obtained some existence criteria for positive solutions of this problem. Motivated by this works, in this paper, our object is to improve the situation. By using the properties of Green function [1, 2] and index fixed point theorem some new existence results for positive solutions.

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are obtained. Moreover, the existence of two positive solutions on the initial value problem (1.1)-(1.2) is also considered.

2. Background materials

Let \( E = C[0, 1] \) be the Banach space with the maximum norm \( ||u|| = \max_{t \in [0, 1]} u(t) \).

We can reduce problem (1.1)-(1.2) to an integral equation in \( E \).

Lemma 2.1. Let \( n \in \mathbb{N}, n - 1 < \beta < \alpha < n \) and assume
\( i: f: [0, 1] \times \mathbb{R} \to \mathbb{R} \) is a continuously differentiable function;
\( ii: f(0, 0) = 0 \) and \( f(t, 0) \neq 0 \) on a compact subinterval of \( 0, 1 \).

Then \( u \in C^n[0, 1] \) is a solution of (1.1)-(1.2) if and only if
\[
  u(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} \eta_k + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v(s) ds, \quad 0 \leq t \leq 1,
\]
where \( v \in C[0, 1] \) is a solution of the equation
\[
  v(t) = \int_0^1 G(t,s) f(s,v(s)) ds, \quad 0 \leq t \leq 1,
\]
and
\[
  G(t,s) = \frac{1}{\Gamma(\alpha - \beta)} (t-s)^{\frac{\alpha - \beta}{2}}. \quad (2.1)
\]

By making use of (2.1) we can prove that \( G(t,s) \) has the following properties.

Proposition 2.2.
\[
  G(t,s) \leq \frac{1}{\Gamma(\alpha - \beta)} t^{\alpha - \beta - 2} (1-s)^{\alpha - \beta - 1}
  \leq \frac{1}{\Gamma(\alpha - \beta)} s^{\alpha - \beta - 2} (1-s)^{\alpha - \beta - 1}, \quad 0 \leq s \leq t \leq 1.
\]

Proposition 2.3. \( G(t,s) \geq \frac{1}{\Gamma(\alpha - \beta)} (1-s)^{\alpha - \beta - 1}, \quad 0 \leq s \leq t \leq 1. \)

Let us list the following assumptions:

(H1): \( f: [0, 1] \times [0, +\infty) \to [0, +\infty) \) is continuous;
(H2): \( \lim \inf_{u \to 0^+} \frac{f(t,u)}{u} > m \) uniformly with respect to \( t \in [0, 1] \);
(H3): \( \lim \sup_{u \to +\infty} \frac{f(t,u)}{u} < k \) uniformly with respect to \( t \in [0, 1] \);
(H4): \( \lim \sup_{u \to 0^+} \frac{f(t,u)}{u} < k \) uniformly with respect to \( t \in [0, 1] \);
(H5): \( \lim \inf_{u \to +\infty} \frac{f(t,u)}{u} > m \) uniformly with respect to \( t \in [0, 1] \);
(H6): There exist \( r_0 > 0 \) such that
\[
f(t, u) < \Gamma(\alpha-\beta) \left\{ \int_0^1 s^{\alpha-\beta-2} (1-s)^{\alpha-\beta-1} \right\}^{-1} r_0, \quad 0 \leq u \leq r_0, \quad 0 \leq t \leq 1;
\]
(2.2)

(H7): There exist \( \bar{r}_0 > 0 \) such that
\[
f(t, u) > \Gamma(\alpha-\beta) \left\{ \int_0^1 (1-s)^{\alpha-\beta-1} \right\}^{-1} \bar{r}_0, \quad 0 \leq u \leq \bar{r}_0, \quad 0 \leq t \leq 1;
\]
(2.3)

where
\[
m > \max\{1, \frac{1}{\Gamma(\alpha-\beta)} \left\{ \int_0^1 (1-s)^{\alpha-\beta-1} ds \right\}\},
\]
(2.4)
\[
k < \left( \frac{1}{\Gamma(\alpha-\beta)} \right) \left\{ \int_0^1 s^{\alpha-\beta-2} (1-s)^{\alpha-\beta-1} ds \right\}^{-1}.
\]
(2.5)

The main tools of the paper is the following well-known fixed point index theorem.

Lemma 2.4. Let \( T : P \to P \) be a completely continuous mapping and \( Ty \neq y \) for \( y \in \partial B_r \). Then we have the following conclusions:

(i): If \( ||y|| \leq ||Ty|| \) for \( y \in \partial B_r \), then \( i(T, B_r, P) = 0 \).

(ii): If \( ||y|| \geq ||Ty|| \) for \( y \in \partial B_r \), then \( i(T, B_r, P) = 1 \).

3. Main Results

Define the operators \( T \) and \( L \) as follows
\[
(Tu)(t) = \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} f(s, u(s)) ds,
\]
(3.1)
\[
(Lu)(t) = \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} u(s) ds.
\]
(3.2)

Theorem 3.1. Suppose that (H1)-(H3) hold. Then the problem (1.1)-(1.2) has at least one positive solution.

Proof. From (H2), there exist \( \epsilon > 0 \) and \( r > 0 \) such that
\[
f(t, u) \geq (m+\epsilon)u, \quad u \in [0, r].
\]
(3.3)

Therefore for all \( u \in \bar{B}_r \cap P \) by (3.3) we have
\[
(Tu)(t) \geq (m+\epsilon) \int_0^t G(t, s) u(s) ds = (m+\epsilon)(Lu)(t), \quad t \in [0, 1].
\]
(3.4)

The operator \( L \), defined in (3.2), is completely continuous and so has a fixed point say, \( \phi^* \), i.e. \( \phi^* = L(\phi^*) \). Now it is easy to show that
\[
u - Tu \neq \mu \phi^*, \quad \forall u \in \partial B_r \cap P, \quad \mu \geq 0,
\]

and therefore it follows that

\[ i(T, B_r \cap P, P) = 0. \]  

(3.5)

By (H3), there exist \( R > r \) such that

\[ f(t, u) \leq (k - \epsilon)u, \quad \forall u \in P, \quad \geq R. \]  

(3.6)

Then for \( u \in \partial B_R \cap P \) by virtue of (3.6) and (2.5), we know that

\[
\|Tu(t)\| \leq k \frac{1}{\Gamma(\alpha - \beta)} \int_0^1 (t - s)^{\alpha - \beta - 1} \|u\|ds
\leq k \frac{1}{\Gamma(\alpha - \beta)} \int_0^1 s^{\alpha - \beta - 2}(1 - s)^{\alpha - \beta - 1}\|u\|ds < R
\]

So, this yields that

\[ i(T, B_R \cap P, P) = 1. \]  

(3.7)

From (3.5) and (3.7) we get

\[ i(T, (B_R \setminus \bar{B}_r) \cap P, P) = i(T, B_R \cap P, P) - i(T, B_r \cap P, P) = 1. \]

Therefore, \( T \) has at least one fixed point on \((B_R \setminus B_r)\). Consequently, problem (1.1)-(1.2) has at least one positive solution. \( \square \)

Similar to Theorem 3.1 we can prove the following theorems.

**Theorem 3.2.** Suppose that (H1),(H4),(H5) are satisfied. Then the problem (1.1)-(1.2) has at least one positive solution.

**Theorem 3.3.** Suppose that (H1),(H2),(H5),(H6) are satisfied. Then the problem (1.1)-(1.2) has at least two positive solutions.

Similarly, we can prove the following theorem.

**Theorem 3.4.** Suppose that (H1),(H3),(H4),(H7) are satisfied. Then the problem (1.1)-(1.2) has at least two positive solutions.

**References**