ADDITIVE MAPS PRESERVING ELEMENTS
ANNIHILATED BY THE POLYNOMIALS $XY - YX^*$
AND $XY + YX^*$

ALI TAGHAVI, FARZANEH KOLIVAND*

Department of Mathematics, Faculty of Mathematical Sciences, Mazandaran University, P. O. Box 47416-1468, Babolsar, Iran;
taghavi@umz.ac.ir; fmath23@yahoo.com

Abstract. Let $H$ be a complex Hilbert space and $B(H)$ denotes the algebra of all bounded linear operators on $H$. Suppose that $\Phi : B(H) \rightarrow B(H)$ is an additive surjective map. We prove that if $\Phi$ satisfies $\Phi(A)\Phi(P) = \lambda \Phi(P)\Phi(A)^*$ if and only if $AP = \lambda PA^*$ for $\lambda \in \{1, -1\}$ and for all $A$ and idempotent $P$ in $B(H)$, then $\Phi$ is a $*$-automorphism.

1. Introduction

Linear preserver problems (LPP) and additive preserver problems are active research topics in operator theory and linear algebra that deal with the characterization of linear or additive maps on matrix spaces and operator algebras that preserve certain properties, subsets, relations or functions (see for example [1], [2], [4], [5]).

Many mathematicians have focused on a kind of preserver problem concerning zeros of polynomials in one or several elements. Particularly commutativity preserver maps have been considered by several authors that means maps preserving zeros of the polynomial $p(x, y) = xy - yx$.

2010 Mathematics Subject Classification. Primary 47B48; Secondary 46L10.
Key words and phrases. Hilbert space, additive map, Zero of polynomials.
* Speaker.
Also there are several papers about zeros of polynomials \( p(x, y) = xy \) and \( p(x, y) = xy + yx \).

Linear or additive maps preserving zeros of \(*\)-polynomial have also been studied by many authors. For example the study of maps preserve the zeros of the \(*\)-polynomial such \( p(x, x^*) = x^*x - xx^* \) and \( p(x, x^*) = x^*x - 1 \) and \( p(x, y^*) = xy^* \) belong to this type of topics.

If \( R \) is a \(*\)-ring, for \( T, S \in R \), denote the operation \( TS \) by \([T, S]_*\), which is a kind of new product. This product is found playing a more important role in some research topics. For example it is closely related to Jordan \(*\)-derivations.

Recently Cui and Hou \cite{3} have characterized the bijective linear maps preserving the zeros of the \(*\)-polynomial \( p(x, x^*, y) = xy - yx^* \) on \( B(H) \) where \( H \) is a complex Hilbert space and \( B(H) \) is the algebra of all bounded linear operators on \( H \). They proved that any such map is of the form \( \Phi(A) = cUAU^* \) where \( c \) is a non-zero scalar in \( \mathbb{R} \) and \( U \) is unitary operator in \( B(H, K) \).

In this paper we want to characterize the form of an additive surjection on \( B(H) \) that preserves the zero of \( p(x, x^*, y) = xy - yx^* \) and \( p(x, x^*, y) = xy + yx^* \) in both directions, such that \( y \) is specifically an idempotent.

2. Main results

The following is an example of a lemma.

**Lemma 2.1.** Let \( \mathcal{A} \) be a factor von Neumann algebra, and \( A \in \mathcal{A} \). Then \( AP = PA^* \) for every projection \( P \in \mathcal{A} \) implies that \( A \in \mathbb{R}I \).

**Lemma 2.2.** Let \( \mathcal{A} \) be a \( C^* \)-algebra and \( A \in \mathcal{A} \). Then \( A^2 = A^*A \) implies that \( A = A^* \).

**Lemma 2.3.** Suppose that \( \mathcal{A}, \mathcal{B} \) are factor von Neumann algebras and let \( \Phi : \mathcal{A} \to \mathcal{B} \) be a surjective map such that for every idempotent \( P \) and every \( A \in \mathcal{A} \) we have \( AP = PA^* \) if and only if \( \Phi(A)\Phi(P) = \Phi(P)\Phi(A)^* \) then:

(a) \( P = 0 \) if and only if \( \Phi(P) = 0 \) for every idempotent \( P \) in \( \mathcal{A} \);

(b) \( \mathbb{R}I \subseteq \Phi(\mathbb{R}I) \).

**Lemma 2.4.** Let \( \mathcal{A} \) and \( \mathcal{B} \) are factor von Neumann algebras and Let \( \Phi : \mathcal{A} \to \mathcal{B} \) be a surjective map which for every idempotent \( P \) and every \( A \in \mathcal{A} \) satisfies these two conditions:

(a) \( AP = PA^* \) if and only if \( \Phi(A)\Phi(P) = \Phi(P)\Phi(A)^* \);

(b) \( AP = -PA^* \) if and only if \( \Phi(A)\Phi(P) = -\Phi(P)\Phi(A)^* \),

then for every projection \( P \) and every \( A \) in \( \mathcal{A} \) we have \( \Phi(P)\Phi(A)^* = 0 \) if and only if \( PA^* = 0 \).
Lemma 2.5. Let $A$ and $B$ be factor von Neumann algebras and $\Phi : A \rightarrow B$ be an additive surjective map which for every idempotent $P$ and every $A \in A$ satisfies two conditions in lemma (2.4). Then

(i) $\Phi(I) = rI$ for some $r \in \mathbb{R}\{0\}$;
(ii) $\Phi$ preserves selfadjoint operators in both directions;
(iii) $\Phi$ preserves skew selfadjoint operators in both directions;
(iv) $\Phi(A^*) = \Phi(A)^*$ for every $A \in A$;
(v) $\Phi(A)^*\Phi(P) = 0$ if and only if $A^*P = 0$ for projection $P$ in $A$;
(vi) $\Phi$ is injective.

Theorem 2.6. Suppose that $\Phi : B(H) \rightarrow B(H)$ is an additive surjection such that $AP = \lambda PA^*$ if and only if $\Phi(A)\Phi(P) = \lambda \Phi(P)\Phi(A)^*$ for $\lambda \in \{1, -1\}$ and every idempotent $P$ and any arbitrary $A \in B(H)$. then $\Phi$ is a $^*$-automorphism.

Proof. Using lemmas (2.1)-(2.5) we claim that $\Phi$ maps projections into projections and orthogonal projections into orthogonal projections. Also $\Phi$ preserves rank-one projections and if $(P_{\alpha})_{\alpha}$ is a maximal family of pairwise orthogonal rank-one projections, then so is $(\Phi(P_{\alpha}))_{\alpha}$. Then we claim that for every $x, y \in H$ there exists an additive function $\tau : \mathbb{C} \rightarrow \mathbb{C}$ such that $\Phi(\lambda x \otimes y) = \tau(\lambda)\Phi(x \otimes y)$ and for the next step we show that $\tau$ is either the identity or the conjugation on $\mathbb{C}$. Finally we prove that there exists a unitary or anti-unitary operator $U$ on $H$ such that

$$\Phi(A) = UAU^*, (A \in F(H)).$$

and we conclude that $\Phi$ is an $^*$-automorphism.

REFERENCES