THE CONTINUITY OF LINEAR AND SUBLINEAR CORRESPONDENCES

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ABSTRACT. We investigate the continuity of linear and sublinear correspondences defined on cones in normed spaces.

1. INTRODUCTION

An investigation of linear correspondences defined on cones in normed spaces was given in [4]. In particular, the existence of a unique iteration semigroup of continuous linear selections of an iteration semigroup of linear correspondences defined on a cone with a finite cone basis is shown in [4]. It is shown in [5] that a regular cosine family consisting of super-additive mappings continuous and homogeneous with respect to positive rationals with compact values has exponential growth. The continuity of a regular cosine family consisting of continuous and additive mappings with compact and convex values defined on cones with nonempty interior in Banach spaces is established in [5]. A generalization of these results in normed spaces can be found in [1]. In this paper, we reintroduce linear and sublinear correspondences on cones in real normed spaces and give some results on continuity. A general form of linear and sublinear correspondences with convex and compact values

2010 Mathematics Subject Classification. Primary 47A06; Secondary 54C60.
Key words and phrases. Linear correspondence, sublinear correspondence, cone.
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is given. We also present some results on invertibility of selections of sublinear correspondences.

We begin with some basic concepts which are needed. A subset $C$ of a real normed space $X$ is a cone if $tC \subseteq C$ for every $t > 0$. A linearly independent set $E$ is said to be a basis of cone $C$ if

$$C = \{x \in X : x = \sum_{i=1}^{n} \lambda_i e_i, n \in Ne_i \in E; \lambda_i \geq 0, i = 1, \cdots, n\}$$

From here we assume that $X$ and $Y$ are two real normed spaces and $C$ is a convex cone of $X$. Let $c(X)$ denote the set of all nonempty and compact subsets of $X$ and $cc(X)$ be the family of all convex sets of $c(X)$. We recall that a correspondence $\varphi$ on any subset $E$ of $X$ is a relation which assigns a nonempty set of $Y$ to each element of $E$. We use the notations $\varphi : C \rightarrow c(Y)$ and $\varphi : C \rightarrow cc(Y)$ for correspondences with compact values and convex and compact values, respectively.

**Definition 1.1.** [4] A correspondence $\varphi : C \rightarrow Y$ is called:

1. linear if $\varphi(x+y) = \varphi(x) + \varphi(y)$ (additivity) and $\varphi(\lambda x) = \lambda \varphi(x)$, for every $x, y \in C$ and $\lambda > 0$.
2. sublinear if $\varphi(x+y) \subseteq \varphi(x) + \varphi(y)$ and $\varphi(\lambda x) = \lambda \varphi(x)$, for every $x, y \in C$ and $\lambda > 0$.

It is clear that every linear correspondence is sublinear but the converse is not true.

**Definition 1.2.** [5] A correspondence $\varphi : C \rightarrow Y$ is said to be bounded if for every bounded subset $E$ of $C$ the subset $\varphi(E)$ is bounded in $Y$.

We recall that a neighborhood of a set $A$ is any set $B$ for which there is an open set $V$ satisfying $A \subseteq V \subseteq B$

**Definition 1.3.** [5] A correspondence $\varphi : C \rightarrow Y$ is said to be:

1. upper semicontinuous at the point $x$ if for every neighborhood $U$ of $\varphi(x)$, there is a neighborhood $V$ of $x$ such that $z \in V$ implies $\varphi(z) \subseteq U$. Also $\varphi$ is upper semicontinuous on $C$, if it is upper semicontinuous at every point of $C$.
2. lower semicontinuous at the point $x$ if for every open set $U$ that $\varphi(x) \cap U \neq \emptyset$ there is a neighborhood $V$ of $x$ such that $z \in V$ implies $\varphi(z) \cap U \neq \emptyset$. $\varphi$ is lower semicontinuous on $C$, if it is lower semicontinuous at every point of $C$.
3. continuous at $x$ if it is both upper and lower semicontinuous at $x$. It is continuous if it is continuous at each point of $C$. 

For each pair of nonempty and compact subsets $A$ and $B$ of $X$, the Hausdorff metric $h$ is defined as

$$h(A, B) = \max\{\sup_{a \in A} d(a; B), \sup_{b \in B} d(b, A)\}$$

where $d(a, B) = \inf_{b \in B} \|a - b\|$. Every correspondence with compact values $\varphi : X \rightarrow Y$ is continuous if and only if $\varphi : X \rightarrow (c(Y), h)$ is continuous in the sense of a single-valued function (see Theorem 17.15 in [3]).

2. Main results

In this section we study the continuity of linear and sublinear correspondences defined on cones with a finite basis in real normed spaces. We start with the following.

**Lemma 2.1.** [5] A sublinear correspondence $\varphi : C \rightarrow Y$ is bounded if and only if there exists a positive constant $M$ such that $\|\varphi(x)\| := \sup\{\|y\| : y \in \varphi(x)\} < M\|x\|, (x \in C)$.


**Lemma 2.2.** [2] Let $0 \in C \subseteq X$. If $\varphi : C \rightarrow Y$ is a bounded-valued sublinear correspondence, then $\varphi$ is upper semicontinuous at zero if and only if $\varphi$ is bounded.

**Theorem 2.3.** [2] Let $E = \{e_1, e_2, \ldots, e_n\}$ be a basis of $C$. If $\varphi : C \rightarrow c(Y)$ is linear, then $\varphi$ is continuous.

**Corollary 2.4.** [2] Let $E = \{e_1, e_2, \ldots, e_n\}$ be a basis of $C$. If $\varphi : C \rightarrow C$ is a linear correspondence, then

$$\varphi(x) = \{l^{-1}Al(x)\}_{A \in M_\varphi}, \quad (0 \neq x \in C)$$

and $\varphi$ is lower semicontinuous at every point.

**Corollary 2.5.** [2] Let $E = \{e_1, e_2, \ldots, e_n\}$ be a basis of $C$. If $\varphi : C \rightarrow c(Y)$ is a sublinear correspondence, then

- $\varphi$ is upper semicontinuous at every point;
- moreover, if $\varphi : C \rightarrow C$, then for every $x \in C \{0\}$ we have

$$\varphi(x) \subseteq \{l^{-1}Al(x)\}_{A \in M_{\text{col}(\varphi)}},$$

where $l$ is the isomorphism given by $l(\sum_{j=1}^{n} \lambda_j e_j) = (\lambda_1, \ldots, \lambda_n)^T$

The following example shows that a sublinear correspondence need not be lower semicontinuous at every point.
Example 2.6. [2] Define $\varphi : [0; +\infty) \times [0; +\infty) \rightarrow [0; +\infty) \times [0; +\infty)$ by

$$\varphi(x, y) = \begin{cases} 
\{(0, 0)\} & x \geq 0, y > 0; \\
\{(t, 0) : 0 \leq t \leq x\} & x \geq 0, y = 0.
\end{cases}$$

It is easy to see that the sublinear correspondence $\varphi$ is not lower semi-continuous at every point $(x, 0)$ where $x > 0$.

For the rest of this section we consider, inspired by [3], the relations between Hausdorff distance of the unit matrix and multimatrix of a linear correspondence and invertibility of its selections. Every cone $C$ with a finite basis $E = \{e_1, e_2, \cdots, e_n\}$ induces a norm on the vector space of all $n \times n$ matrices $M_n(\mathbb{R})$ by

$$\|A\| = \sup\{\|\sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_j a_{ij} e_i\| : \sum_{j=1}^{n} \lambda_j e_j \in C, \|\sum_{j=1}^{n} \lambda_j e_j\| = 1\}, \quad (2.1)$$

for every $A = [a_{ij}]$ (see [3]). In the following, $h_1$ and $I$ will denote the Hausdorff metric derived from the norm given in (2.1) and the unit matrix, respectively.

Lemma 2.7. [2] Suppose that $C$ has a finite cone basis. If $\varphi : C \rightarrow c(C)$ is a linear correspondence, then

$$h_1(M_\varphi, \{I\}) = \sup\{h(\varphi(x), \{x\}) : x \in C, \|x\| = 1\}.$$

Corollary 2.8. [2] Suppose that $C$ has a finite cone basis. If $\varphi : C \rightarrow c(C)$ is a sublinear correspondence, then

$$h_1(M_\varphi, \{I\}) \geq \sup\{h(\varphi(x), \{x\}) : x \in C, \|x\| = 1\},$$

where $\hat{\varphi}$ is given by $\hat{\varphi}(x) = \sum_{j=1}^{n} \lambda_j \hat{c}(\varphi(e_j))$.

Corollary 2.9. [2] Let $\{e_1, e_2, \cdots, e_n\}$ be a finite basis of $C$. Then, there exists an $\eta > 0$ such that for every sublinear correspondence $\varphi : C \rightarrow c(C)$ satisfying $h_1(M_{\text{co}(\varphi)}, \{I\}) < \eta$, each $A \in M_\varphi$ is invertible.

References