EXISTENCE OF BOGDANOV-TAKENS BIFURCATION IN A FOUR-NEURON NEURAL NETWORK WITH DELAYS

ZOHREH DADI*

Department of Mathematics, University of Bojnord, Bojnord, I.R.Iran

Abstract. In this paper, a four-neuron ring with self-feedback and delays is considered. This network is a nonlinear system of ordinary differential equations with time delays. The main purpose of this paper is to obtain necessary conditions on parameters of the model which guarantee existence of Bogdanov-Takens bifurcation.

1. Introduction

During the last two decades, there has been an increasing interesting in study the dynamical neural network models with time delay. In fact, in neural networks, the delay occurs in the signal transmission between neurons or electronic model neurons due to finite propagation velocity of action potentials. Delays make neural networks more useful and versatile, therefore there are many results about the local and global stability for neural network models and periodic solutions such as [1, 2, 3] and the references cited therein. In fact, there have been published a large amount of papers on the dynamical behavior (stability, Hopf bifurcation) of the neural networks, only a few can be found on the neural networks which is studied from the viewpoint of Bogdanov-Takens bifurcation. This is the motivation of the present
paper. In this paper, we consider the following system

\begin{align*}
\dot{x}_1(t) &= -r_1 x_1(t) + g_1(x_1(t)) + f_1(x_4(t - \tau_2)) + f_1(x_2(t - \tau_2)) \\
\dot{x}_2(t) &= -r_2 x_2(t) + g_2(x_2(t)) + f_2(x_1(t - \tau_1)) + f_2(x_3(t - \tau_1)) \\
\dot{x}_3(t) &= -r_3 x_3(t) + g_3(x_3(t)) + f_3(x_4(t - \tau_2)) + f_3(x_2(t - \tau_2)) \\
\dot{x}_4(t) &= -r_4 x_4(t) + g_4(x_4(t)) + f_4(x_1(t - \tau_1)) + f_4(x_3(t - \tau_1)).
\end{align*}

where \(x_i(t), i = 1, 2, 3, 4\) represents the state of the \(i\)-th neuron at time \(t\), \(r_i > 0\) is the internal decay rate, \(f_i\) is the connection function between of neurons, \(g_i\) represents the nonlinear feedback function, \(\tau_i \geq 0\) is the connection time delay. The main property of this model is bidirectional with two loops. This system can model the evolution of a Hopfield-Cohen-Grossberg network consisting of four elements with time delayed nearest-neighbour coupling, as illustrated schematically in Fig. 1. The stability and Hopf bifurcation (globally) of this model are studied in [1, 3]. Therefore, the study carried out in the present paper may contribute to understand the Bogdanov-Takens singularity in 4 cells with time delays. To the best of our knowledge, we only studied this type of architecture of this network from the viewpoint of existence of Bogdanov-Takens bifurcation. By investigating of the distribution of roots of the characteristic equation of the linear part of system 1.1 at the equilibrium point, we determine conditions on the parameters of nonlinear system 1.1 guaranteeing the existence of the Bogdanov-Takens bifurcation.

2. Main results

To establish the main results for system 1.1, it is necessary to make the following assumption

\((\text{H1}) \ f_i(0) = g_i(0) = 0 \) for \(i = 1, 2, 3, 4\),
(H2) $\tau_1 + \tau_2 = \tau$.

Also, let

$$
\begin{align*}
    u_1(t) &= x_1(t - \tau_1), \\
    u_2(t) &= x_2(t), \\
    u_3(t) &= x_3(t - \tau_1), \\
    u_4(t) &= x_4(t).
\end{align*}
$$

Then, by (H2), we have the following system

$$
\begin{align*}
    \dot{u}_1(t) &= -r_1 u_1(t) + g_1(u_1(t)) + f_1(u_4(t - \tau)) + f_1(u_2(t - \tau)), \\
    \dot{u}_2(t) &= -r_2 u_2(t) + g_2(u_2(t)) + f_2(u_1(t)) + f_2(u_3(t)), \\
    \dot{u}_3(t) &= -r_3 u_3(t) + g_3(u_3(t)) + f_3(u_4(t - \tau)) + f_3(u_2(t - \tau)), \\
    \dot{u}_4(t) &= -r_4 u_4(t) + g_4(u_4(t)) + f_4(u_1(t)) + f_4(u_3(t)).
\end{align*}
$$

By (H1), it is obvious that the origin is trivial solution of system 2.1. The characteristic equation of system 2.1 is as follows,

$$
P(\lambda, \tau) = \lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d + (m\lambda^2 + n\lambda + r)e^{-\lambda\tau} \quad (2.2)
$$

where

$$
\begin{align*}
    l_i &= r_i - g'_i(0), \quad i = 1, 2, 3, 4 \\
    a &= \sum_{i=1}^{4} l_i \\
    b &= l_1(\sum_{i=2}^{4} l_i) + l_2(\sum_{i=3}^{4} l_i) + l_3 l_4 \\
    c &= l_1 l_2(\sum_{i=3}^{4} l_i) + l_4 l_3(\sum_{i=1}^{2} l_i) \\
    d &= \prod_{i=1}^{4} l_i \\
    m &= (f'_1 + f'_3)(f'_4 - f'_2) \\
    n &= (l_1 + l_2)f'_3 f'_4 + (l_2 + l_3)f'_1 f'_4 - (l_1 + l_4)f'_3 f'_2 - (l_3 + l_4)f'_1 f'_2 \\
    r &= l_1 l_2 f'_3 f'_4 + l_3 l_2 f'_1 f'_4 - l_1 l_4 f'_3 f'_2 - l_3 l_4 f'_1 f'_2.
\end{align*}
$$
Theorem 2.1. The equation 2.2 is stable for \( \tau \geq 0 \) if and only if
\[ a > 0 \quad c + n > 0 \quad r + d > 0 \quad a(b + m)(c + n) > (c + n)^2 + a^2(r + d) \]  
(2.3)

Proof. It is clear, by Routh-Hurwitz criteria. \( \square \)

Now we establish conditions under which Eq.2.2 has double zero singularity.

Theorem 2.2. The Bogdanov-Takens bifurcation can be occur in system 2.1 if one of the following conditions holds;

A) \[ d \neq 0, \quad r = -d, \quad \tau = \frac{c + n}{-d}, \quad \frac{b + m}{c + n} \neq \frac{c + 3n}{2d}, \]  
(2.4)

B) \[ d = 0, \quad r = 0, \quad n = -c, \quad m \neq -\tau c - b \]  
(2.5)

Proof. It is sufficient to investigate the existence of double zero singularity in Eq.2.2. \( \square \)

References


E-mail address: z.dadi@ub.ac.ir