THE BIRKHOFF-JAMES ORTHOGONALITY AND NORM PARALLELISM FOR THE $p$-SCHATTEN CLASS

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ABSTRACT. Utilizing the Birkhoff–James orthogonality, we present a characterization of the norm parallelism for the trace-class operators on a finite dimensional Hilbert space. In addition, we consider the norm parallelism problem for the $p$-Schatten class.

1. INTRODUCTION

The most commonly used definition of orthogonality in normed linear spaces, with the norm not necessarily coming from an inner product, is the Birkhoff–James orthogonality [3]: if $x$ and $y$ are elements of a normed linear space $(X, \| \cdot \|)$, then $x$ is orthogonal to $y$ in the Birkhoff–James sense, in short $x \perp_B y$, if

$$\| x + \mu y \| \geq \| x \|, \quad (\mu \in \mathbb{C}).$$

It is easy to see that in an inner product space the Birkhoff–James orthogonality becomes the usual one.

When $X$ is a Hilbert $C^*$-module, some interesting characterizations of Birkhoff–James orthogonality were given in [1, 2].

Recall that, an element $x \in X$ is said to be the norm parallel to another element $y \in X$, denoted by $x \parallel y$ if

$$\| x + \lambda y \| = \| x \| + \| y \|$$

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for some $\lambda \in \mathbb{T} = \{\alpha \in \mathbb{C} : |\alpha| = 1\}$; see [4].

In the case of inner product spaces the norm parallel relation is exactly the usual vectorial parallel relation, that is, $x \parallel y$ if and only if $x$ and $y$ are linearly dependent. In the case of normed linear spaces two linearly dependent vectors are norm parallel, but the converse is false in general.

Some characterizations of the norm parallelism for elements of $C^*$-algebras and Hilbert $C^*$-modules were given in [4, 5].

Next we define the von Neumann-Schatten classes $C_p$ ($1 \leq p < \infty$). Let $B(H)$ denote the algebra of all bounded linear operators on a complex separable Hilbert space $H$ and let $T \in B(H)$ be compact, and let $s_1(T) \geq s_2(T) \geq \cdots \geq 0$ denote the singular values of $T$, i.e., the eigenvalues of $|T| = (T^*T)^{\frac{1}{2}}$ arranged in their decreasing order. The operator $T$ is said to be belong to the Schatten $p$-classes $C_p$ if

$$\|T\|_p = \left[ \sum_{i=0}^{\infty} s_i(T)^p \right]^{\frac{1}{p}} = \left[ \text{tr}(T^p) \right]^{\frac{1}{p}} \quad 1 \leq p < \infty,$$

where tr denotes the trace functional. Hence $C_1$ is the trace class and $C_2$ is the Hilbert-Schmidt class.

The main purpose of this talk is to characterize the Birkhoff-James orthogonality and norm parallelism in $C_p$. Throughout this paper we assume that $H$ is a finite dimensional Hilbert space.

2. Main results

The following theorem characterizes the Birkhoff-James orthogonality in the space $C_1$.

**Theorem 2.1.** Let $T, S \in C_1$. Let $T$ be invertible and let $T = U|T|$ be the polar decomposition of $T$. Then the following statements are equivalent:

(i) $T \perp_B S$.

(ii) $\text{tr}(U^*S) = 0$.

Now we present a characterization of norm parallelism for the trace-class operators.

**Theorem 2.2.** Let $T, S \in C_1$ be invertible and $T = U|T|, S = V|S|$ be their polar decompositions. Then the following statements are equivalent:

(i) $T \parallel S$. 

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(ii) \( \|T\|_1 \left| \text{tr}(U^*S) \right| = \|S\|_1 \text{tr}(|T|) \).

(iii) \( \|S\|_1 \left| \text{tr}(V^*T) \right| = \|T\|_1 \text{tr}(|S|) \).

In the next theorem we characterize the Birkhoff–James orthogonality in the space \( C_p (1 < p < \infty) \).

**Theorem 2.3.** Let \( T, S \in C_p \) and let \( T = U|T| \) be the polar decomposition of \( T \). Then the following statements are equivalent:

(i) \( T \perp_B S \).

(ii) \( \text{tr}(|T|^{p-1}U^*S) = 0 \).

We finish this section with a characterization of the norm parallelism for the \( p \)-Schatten class.

**Theorem 2.4.** Let \( T, S \in C_p \) and let \( T = U|T|, S = V|S| \) be their polar decompositions. Then the following statements are equivalent:

(i) \( T \parallel S \).

(ii) \( \|T\|_p \left| \text{tr}(|T|^{p-1}U^*S) \right| = \|S\|_p \text{tr}(|T|^p) \).

(iii) \( \|S\|_p \left| \text{tr}(|S|^{p-1}V^*T) \right| = \|T\|_p \text{tr}(|S|^p) \).

**References**